Homework 5

Problem 1 (Chapter 8, P. 22)

$T_0 = 2$, $f_0 = \frac{1}{2}$, $\omega_0 = \pi$,

$x(t) = \begin{cases} \sin(2\pi t), & |t| < 1/2 \\ 0, & 1/2 < |t| < 1 \end{cases}$, as shown in figures (dotted line).

Find the harmonic function $X[k]$ by:

$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} \sin(2\pi t)e^{-jk\pi t}dt = \frac{1}{2}\int_{-1/2}^{1/2} \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} e^{-jk\pi t}dt$$

$$= \frac{1}{4j}\int_{-1/2}^{1/2} (e^{j(2-k)\pi t} - e^{-j(2+k)\pi t})dt = \frac{1}{4j}\left[\left.\frac{e^{j(2-k)\pi t}}{j\pi(2-k)}\right|_{-1/2}^{1/2} + \left.\frac{e^{-j(2+k)\pi t}}{j\pi(2+k)}\right|_{-1/2}^{1/2} \right]$$

$$= \frac{1}{2j}\left[\frac{\sin(\frac{\pi}{2}(2-k))}{\pi(2-k)} - \frac{\sin(\frac{\pi}{2}(2+k))}{\pi(2+k)} \right]$$

$$= -\frac{j}{4}\left[\text{sin c}(\frac{2-k}{2})\text{-sinc}(\frac{2+k}{2}) \right]$$

So, the complex CTFS description is:

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k]e^{jk\pi t}$$

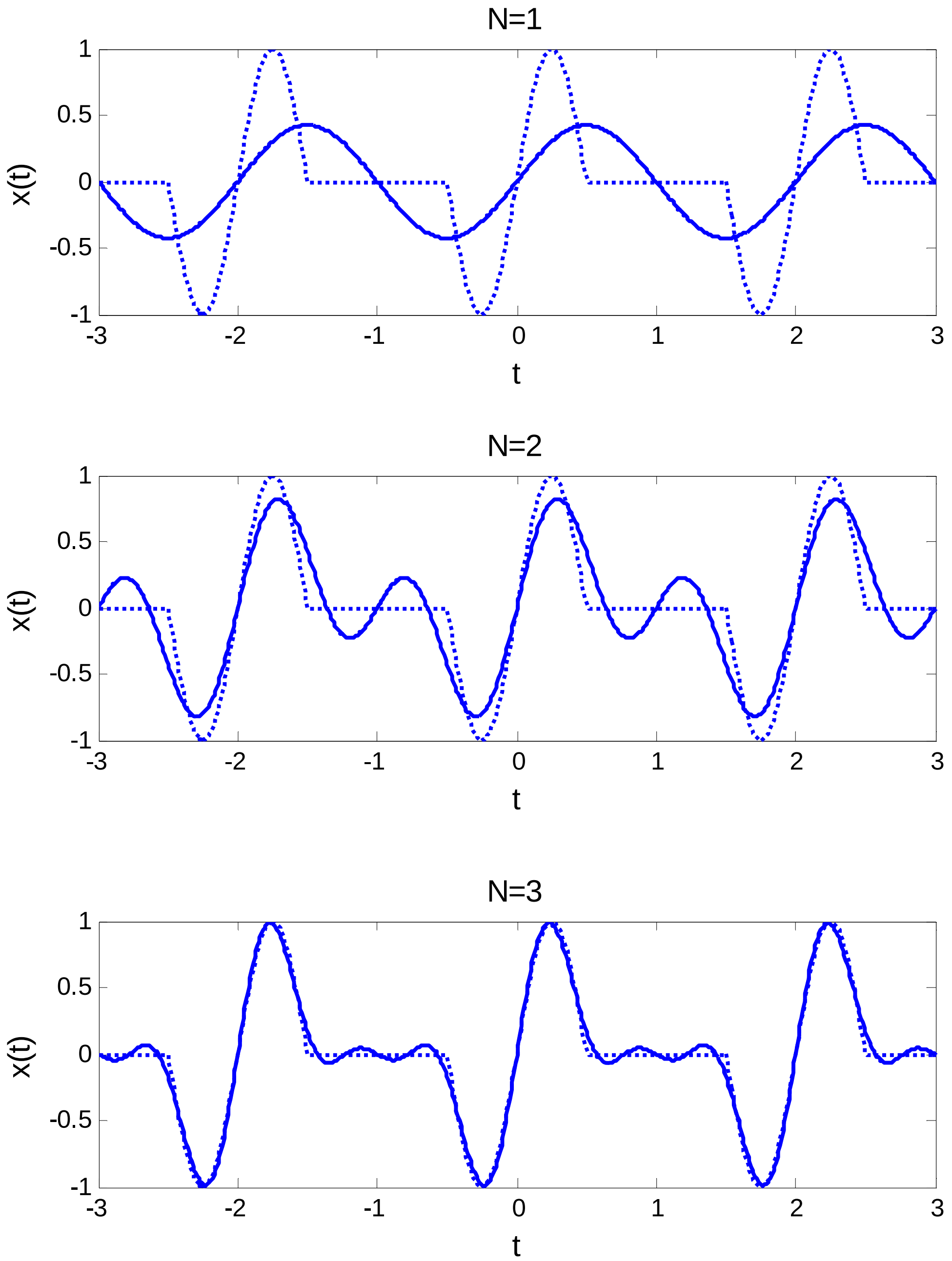
Approximation to the signal:

$$x_N(t) = \sum_{k=-N}^{k=N} X[k]e^{jk\pi t}$$, as shown is figures (solid line)

Code:

```
clear all
T0 = 2;
icount = 0;
for t = -3:0.01:3;                                  % To get x(t)
    icount = icount+1;
    if abs(t-round(t/T0)*T0) <= 1/2;
        x_t(icount) = sin(2*pi*t);
```

```
        else x_t(icount) = 0;
        end
end
tt = -3:0.01:3;
figure(1);subplot(2,1,1),plot(tt,x_t,':');hold on
x_Nt = zeros(size(t));                                  % To get x[k]
N = 1;                                                  % N=1,2,3
for k = 1:N;
        X_k1(k) = -j/4*(sinc((2-k)/2)-sinc((2+k)/2));
        X_k2(k) = -j/4*(sinc((2+k)/2)-sinc((2-k)/2));
        x_Nt = x_Nt+X_k1(k)*exp(j*2*pi*k*1/T0*tt)++X_k2(k)*exp(-j*2*pi*k*1/T0*tt);
end
figure(1);subplot(2,1,1),plot(tt,x_Nt);hold off
```



Problem 2 (Chapter 8, P. 23)

From (a) to (d), we got

$$x_N(t) = x_{N-1}(t) + 2\cos(2N\pi t)$$

$$= x_{N-2}(t) + 2\cos(2(N-1)\pi t) + 2\cos(2N\pi t)$$

$$\vdots$$

$$= x_0(t) + 2\cos(2\pi t) + 2\cos(4\pi t) + \cdots + 2\cos(2N\pi t)$$

$$= 1 + 2\sum_{n=1}^{N}\cos(2n\pi t) = \sum_{n=-N}^{N} e^{j2n\pi t}$$

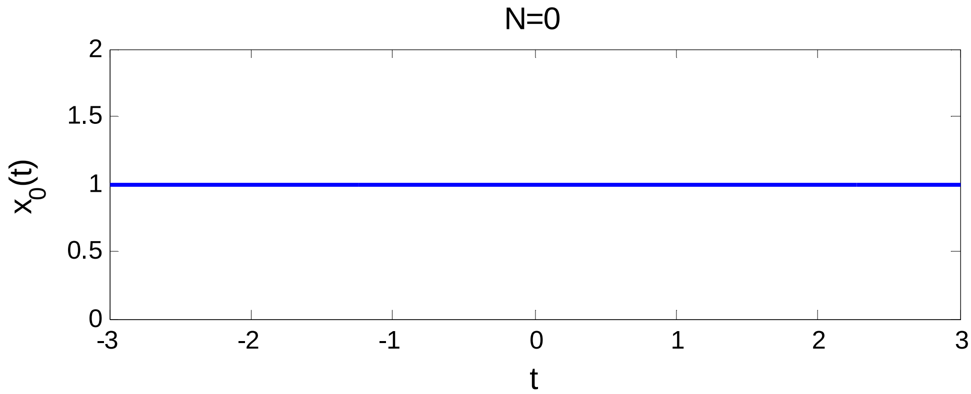
Signals for $N = 0, 1, 2, 20$ are shown as follow over the time range $-3 < t < 3$. By numerically calculate the area of the signal over $-1/2 < t < 1/2$, we got:

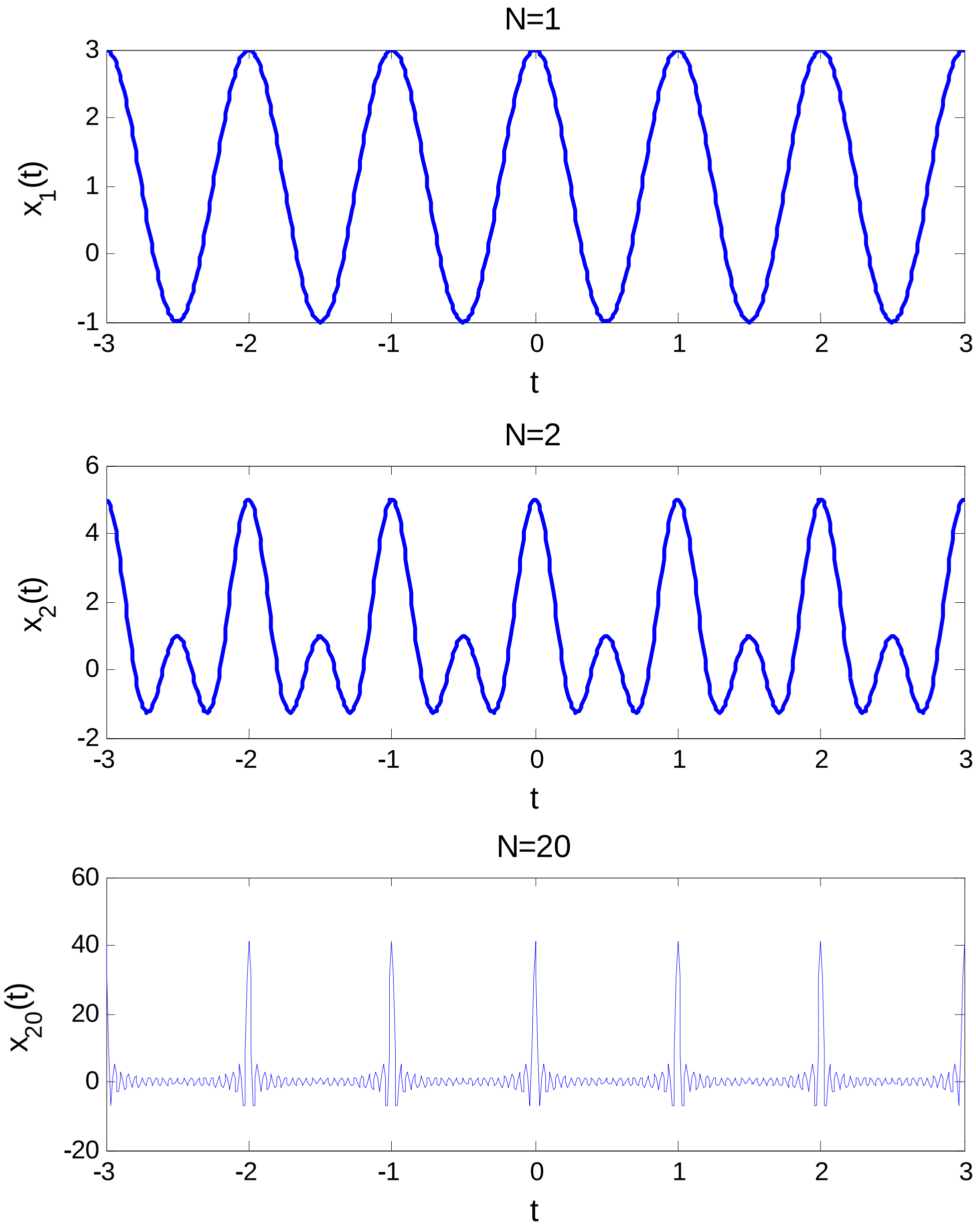
$(a)\ N = 0,\ area = 1,\ \ (b)\ N = 1,\ area = 1,\ \ (c)\ N = 2,\ area = 1,\ \ (d)\ N = 20,\ area = 1$

As N become lager, for case $N = \infty$, $x_N(t) = \sum_{n=-\infty}^{\infty} e^{j2n\pi t}$ tends to a function $x(t)$ with $T_0 = 1$ and $X[k] = 1$. From the table we can find that, $x(t)$ is a unit period impulse ($x(t) = \delta(t) \leftrightarrow X[k] = 1$). Clearly see in plots, as N increases the figure become more like the impulse train and the area within one period should be 1.

Code:

```
t = -3:0.01:3; x_Nt = zeros(size(t))+1; N0 = 20;        % N0=0,1,2,20 for part a,b,c,d
for n = 1:N0;
    x_Nt = x_Nt+2*cos(2*n*pi*t);
end
figure(2),subplot(2,1,1),plot(t,x_Nt);
num1 = find(t==-1/2);num2 = find(t==1/2);
Area = sum((x_Nt(num1:num2-1))*0.01,        % Summation should end at num2-1
```





Problem 3 (Chapter 8, P. 30)

(1) From the first figure in Figure E.30, the signal is even with $T_0 = 2, \omega_0 = \pi$

$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_{-1}^{1} x(t)[\cos(-k\pi t) + j\sin(-k\pi t)]dt$$

$$= \frac{1}{2}\int_{-1}^{1} \underbrace{x(t)\cos(k\pi t)}_{even} dt - \frac{j}{2}\int_{-1}^{1} \underbrace{x(t)\sin(k\pi t)}_{odd} dt$$

So, $\frac{j}{2}\int_{-1}^{1} x(t)\sin(k\pi t)dt = 0$, the harmonic function $X[k]$ have a purely real value for every value of k.

(2) From the second figure in Figure E.30, the signal is odd with $T_0 = 1, \omega_0 = 2\pi$

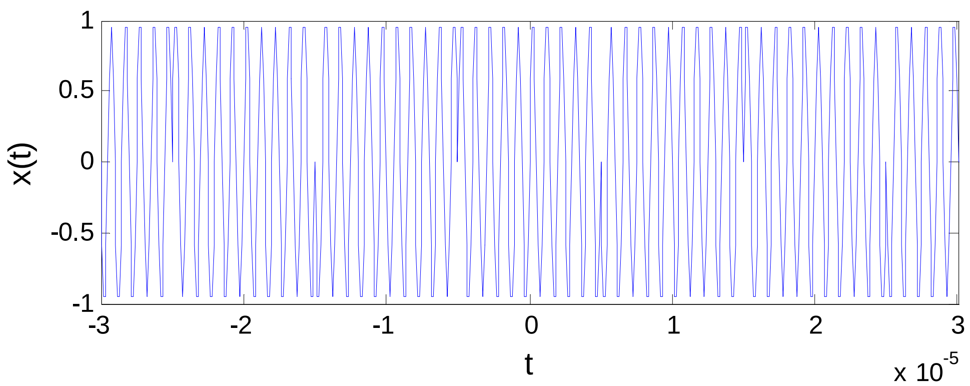
$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \int_{-1/2}^{1/2} x(t)e^{-jk2\pi t}dt$$

$$= \int_{-1/2}^{1/2} \underbrace{x(t)\cos(2k\pi t)}_{odd} dt - j\int_{-1/2}^{1/2} \underbrace{x(t)\sin(2k\pi t)}_{even} dt$$

So, $\int_{-1/2}^{1/2} x(t)\cos(2k\pi t)dt = 0$, the harmonic function $X[k]$ have a purely imaginary value for every value of k.

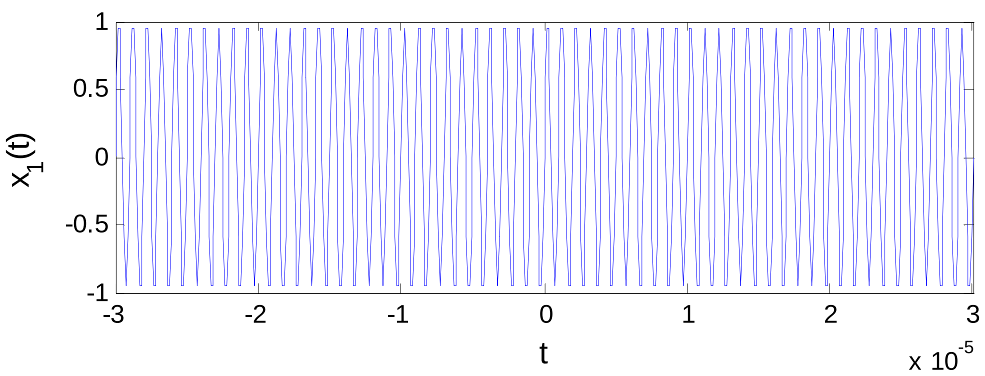
Problem 4 (Chapter 8, P. 32)

Period of a sine wave is $\frac{1}{10^6}$; period of a burst of "1" is $\frac{1}{10^5}$; period of a burst of "0" is $\frac{1}{10^5}$; period of a binary signal with alternating 1's and 0's is $T_0 = 2\times\frac{1}{10^5}$ and $\omega_0 = \pi\times 10^5$. The binary signal we want should be like:

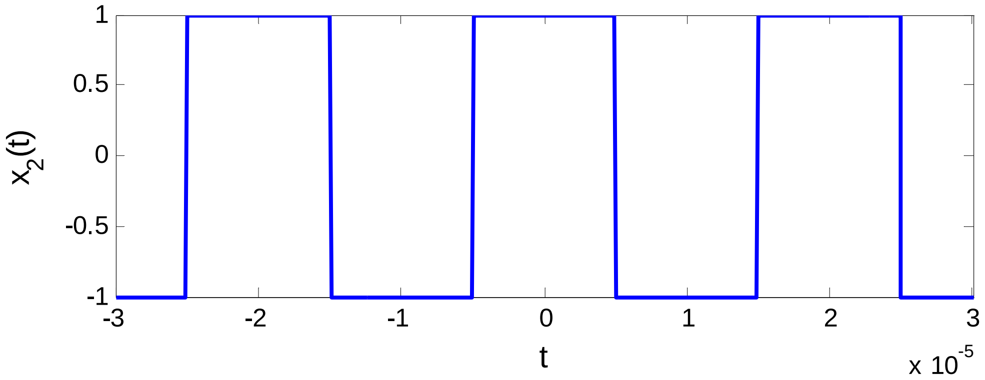


How can it come?

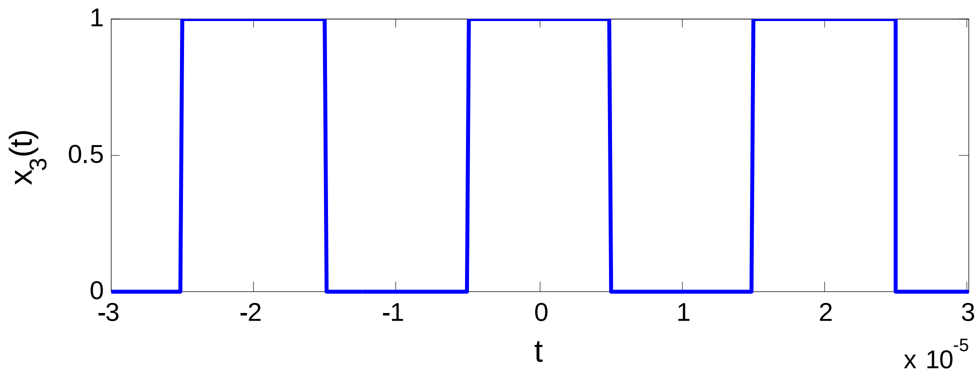
As we already know the basic sine wave is $x_1(t) = \sin(2\times 10^6 \pi t)$, shown as



In order to get the wanted signal, the signal shown above should multiply with a square wave signal $x_2(t)$ looks like:



How to get this square wave signal? It's just from a similar and basic function $x_3(t)$, as shown below. Then $x_2(t) = x_3(t) \times 2 - 1$.



The expression for this $x_3(t)$ is just a convolution of a rectangle function and impulse train:

$$x_3(t) = \text{rect}\left(\frac{t}{1/10^5}\right) * \delta_{2\times 1/10^5}(t)$$

$$\Rightarrow \quad x_2(t) = 2x_3(t) - 1 = 2\text{rect}\left(\frac{t}{1/10^5}\right) * \delta_{2\times 1/10^5}(t) - 1$$

$$x(t) = x_1(t)x_2(t) = \sin(2\times 10^6 \pi t) \times \left(2\text{rect}\left(\frac{t}{1/10^5}\right) * \delta_{2\times 1/10^5}(t) - 1\right)$$

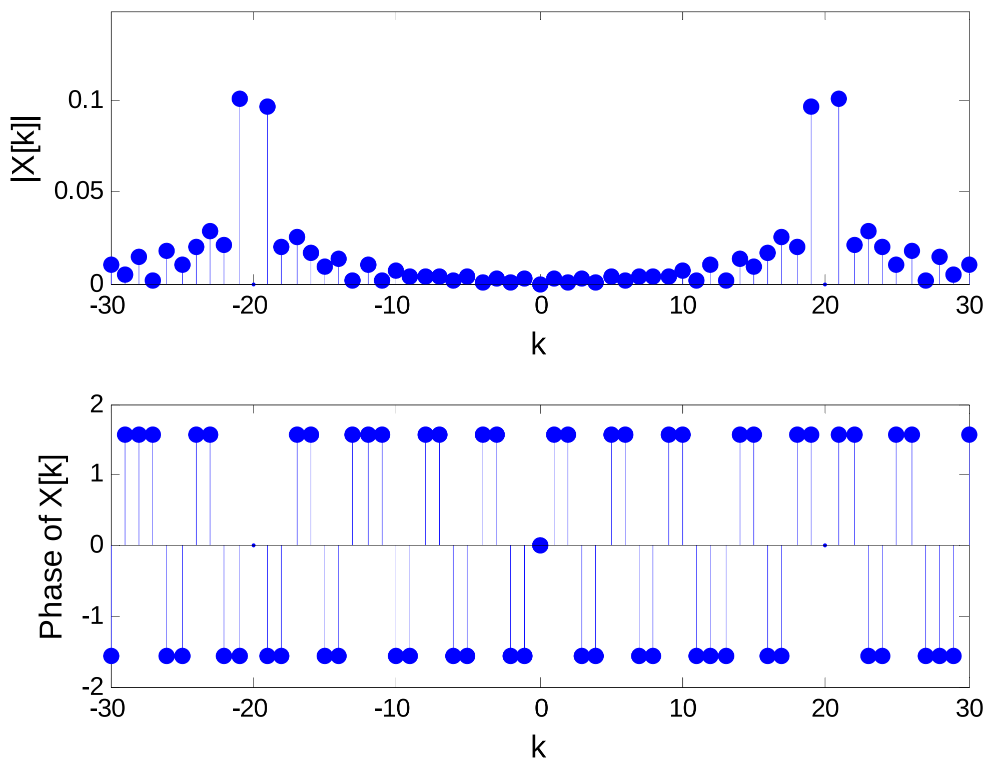
From Fourier Series Pairs table, with $T_0 = 2 \times \frac{1}{10^5}$, $\omega_0 = \pi \times 10^5$, width of rectangular function is $T = \frac{1}{10^5}$,

$$X[k]=\frac{1}{2j}\left[\delta[k-20]-\delta[k+20]\right]*\left[2\times\frac{1/10^5}{2/10^5}\text{sinc}\left(\frac{1/10^5\times k\pi\times 10^5}{2}\right)-\delta[k]\right]$$

$$=\frac{1}{2j}\left[\delta[k-20]-\delta[k+20]\right]*\left[\text{sinc}\left(\frac{k\pi}{2}\right)-\delta[k]\right]$$

$$=-\frac{j}{2}\left[\text{sinc}\left(\frac{(k-20)\pi}{2}\right)-\text{sinc}\left(\frac{(k+20)\pi}{2}\right)-\delta[k-20]+\delta[k+20]\right]$$

The magnitude and phase of the harmonic function are shown as follow:



Code:

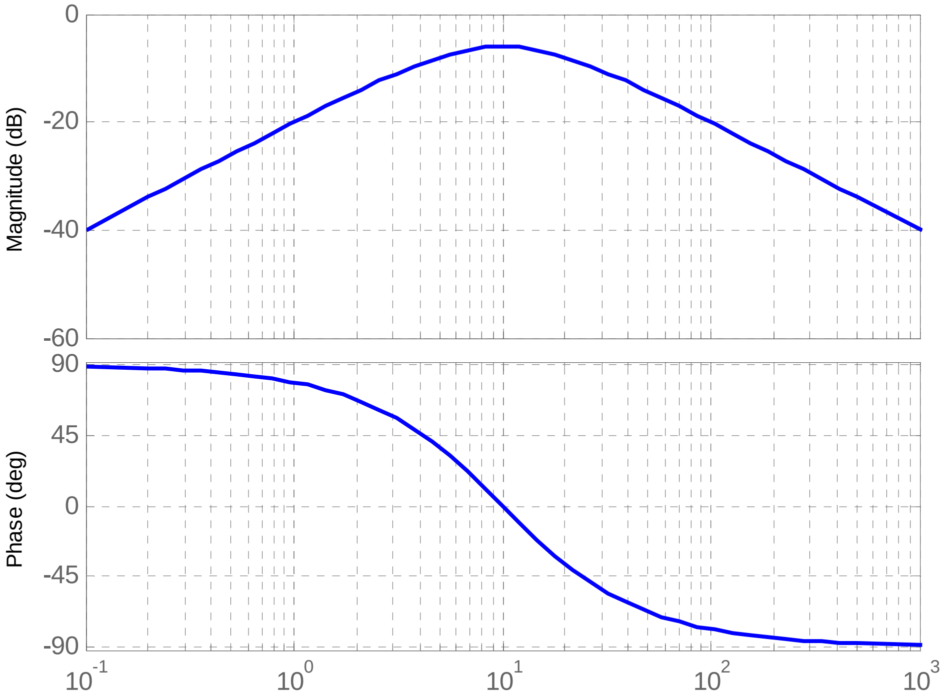
```
k = -30:30;
N_k = 1/(2*j).*(sinc(pi/2.*(k-20))-sinc(pi/2.*(k+20))-dirac(k-20)+dirac(k+20));
figure(1), subplot(2,1,1),stem(k,abs(N_k),'fill'),
figure(1), subplot(2,1,2),stem(k,angle(N_k),'fill'),
```

Problem 5 (Chapter 12, P. 32)

(b). $H(j\omega)=\frac{10}{j\omega+10}\cdot\frac{j\omega}{j\omega+10}=\frac{10j\omega}{(j\omega)^2+20j\omega+100}$

Code:

```
omg = 0.1:0.1:1000;
num = [10,0];den = [1,20,100];
sys = tf(num,den);
figure(1),bode(omg,sys);grid on;                    % using "bode" to get the Bode diagram
```



Asymptote:

$$H(j\omega)=\frac{j10\omega}{(j\omega+10)^2}=\frac{10}{j\omega+10}\cdot\frac{j\omega}{j\omega+10}$$

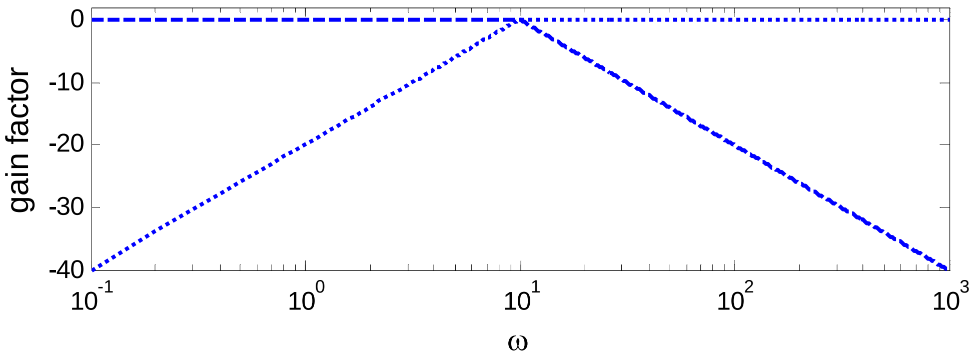
From the expression, we see the frequency response have: zero at $j\omega=0$, pole at $j\omega=-10$ and pole at $j\omega=-10$.

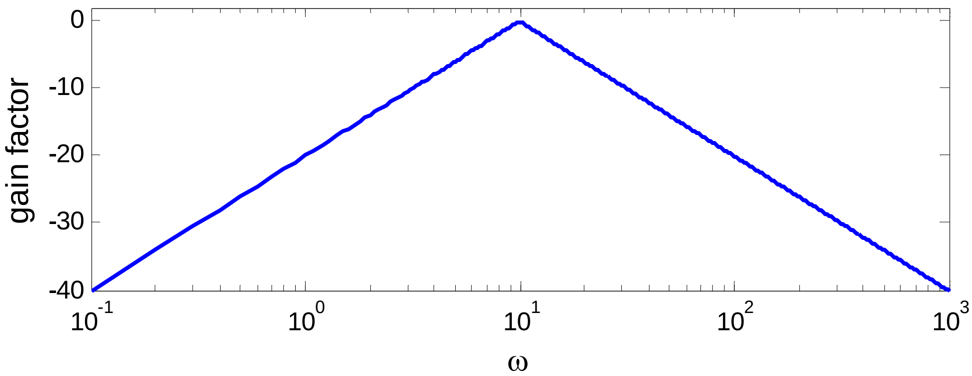
(1). Magnitude asymptote

First consider the part $\frac{10}{j\omega+10}$. For low frequencies ($\omega \ll \omega_p$), the dB-scale magnitude is $20\log_{10}\left|\frac{10}{10}\right|=0$. For high frequencies ($\omega \gg \omega_p$), magnitude should be $20\log_{10}\left|\frac{10}{j\omega}\right|$, which is a straight line with the slope of -20dB and go through (10,0). The magnitude asymptote from $\frac{10}{j\omega+10}$ is a pair of straight lines with corner frequency at $\omega_p=10$, as shown (dashed line).

Similarly, for part $\frac{j\omega}{j\omega+10}$, $20\log_{10}\left|\frac{j\omega}{j\omega+10}\right| \approx 20\log_{10}\left|\frac{j\omega}{10}\right|$ at low frequencies ($\omega \ll \omega_p$), which is a straight line with the slope of 20dB and go through (10,0). At high frequencies ($\omega \gg \omega_p$), $20\log_{10}\left|\frac{j\omega}{j\omega+10}\right| \approx 20\log_{10}\left|\frac{j\omega}{j\omega}\right| = 0$. So, the magnitude asymptote from $\frac{j\omega}{j\omega+10}$ is also a pair of straight lines intersect at $\omega = \omega_p$, as shown (dotted line).

At last, applying compose the dashed line and dotted line to get the final magnitude asymptote, as shown (solid line).





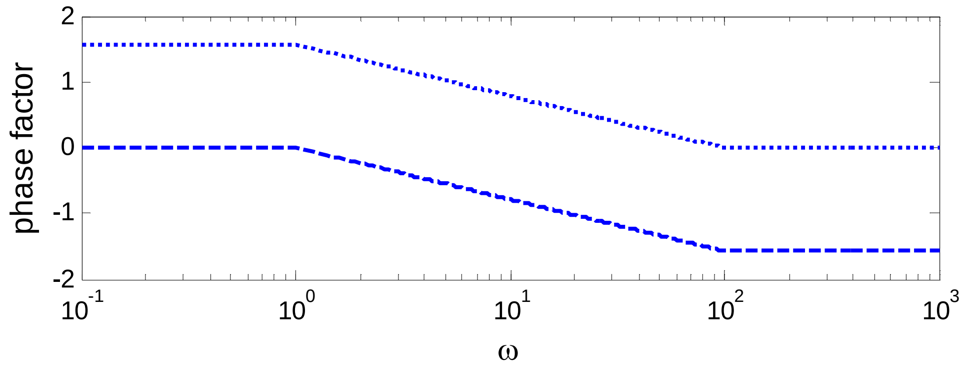
(2). Phase asymptote

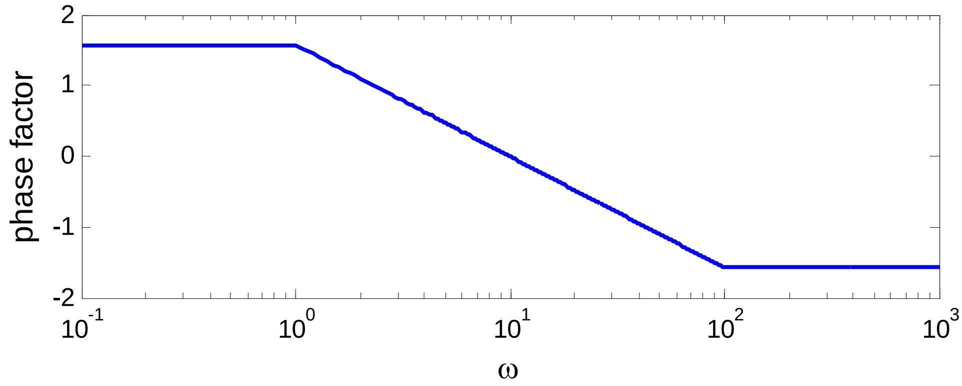
The phase asymptote from $\frac{10}{j\omega+10}$ contains three segments: 0 for low frequencies ($\omega \ll \omega_p/10$) and $-\frac{\pi}{2}$ for high frequencies ($\omega \gg 10\omega_p$), and at frequencies around

$\omega_p = 10$, a straight line (reference to $\log_{10}\omega$) achieves the $-\frac{\pi}{2}$ phase change, as shown (dashed line).

Similarly, for part $\frac{j\omega}{j\omega+10}$, $\frac{\pi}{2}$ for low frequencies ($\omega \ll \omega_p/10$) and 0 for high frequencies ($\omega \gg 10\omega_p$), a straight line at $\omega_p/10 < \omega < 10\omega_p$ gives the $-\frac{\pi}{2}$ phase change, as shown (dotted line).

At last, applying composition rules to get the final phase asymptote, as shown (solid line).





Code:

```
omg = 0.1:0.1:1000;
omg_p = 10;
omg1 = 0.1:0.1:omg_p;
omg2 = omg_p+0.1:0.1:1000;
ym = zeros(size(omg));
ym_1 = ones(size(omg1));                % get the magnitude asymptote for low frequency
ym_2 = omg_p./omg2;                     % get the magnitude asymptote for high frequency
ym(1:length(omg1)) = ym_1;
ym(length(omg1)+1:end)= ym_2;
```

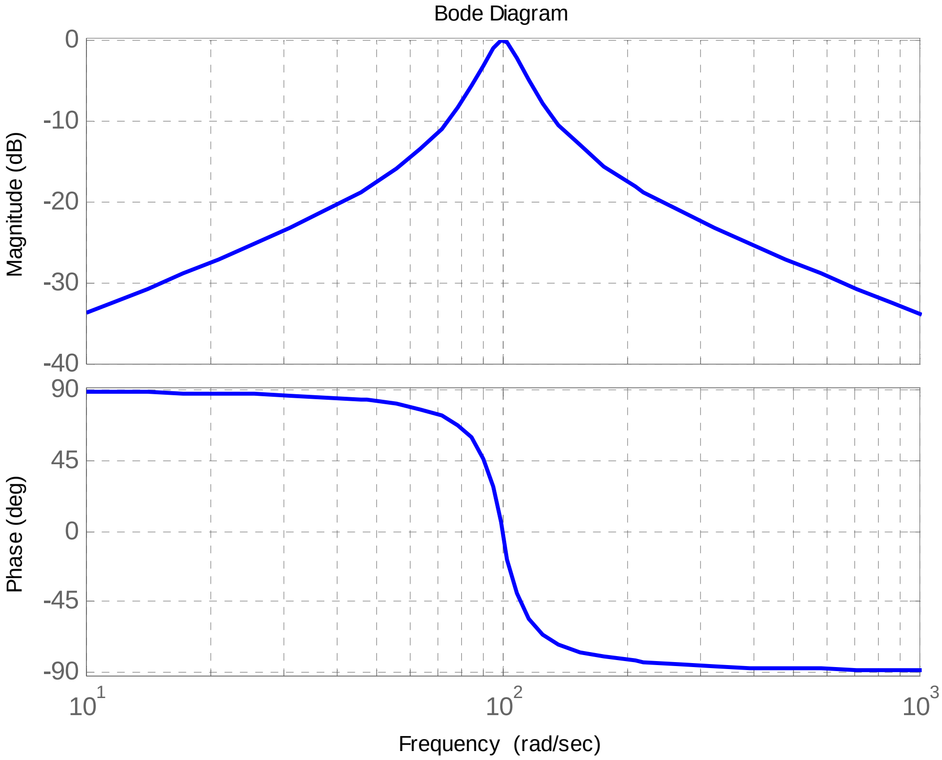
```
asym_1 = 20*log10(ym);
figure(2),subplot(2,1,1),semilogx(omg,asym_1,'--'),hold on
xlabel('\omega'),ylabel('gain factor')
zm = zeros(size(omg));
zm_1 = omg1/omg_p;                 % get the magnitude asymptote from the pole for low frequency
zm_2 = ones(size(omg2));           % get the magnitude asymptote from the pole for high frequency
zm(1:length(omg1)) = zm_1;
zm(length(omg1)+1:end) = zm_2;
asym_2 = 20*log10(zm);
figure(2),subplot(2,1,1),semilogx(omg,asym_2,':'),hold off

omg3 = 0.1:0.1:omg_p/10;
omg4 = omg_p/10+0.1:0.1:omg_p*10;
omg5 = omg_p*10+0.1:0.1:1000;
yp = zeros(size(omg));
yp_1 = zeros(size(omg3));                                   % phase is zero for low frequency
yp_2 = -1*pi/2/2*(log10(omg4)-log10(omg_p/10));
% a -pi/2 phase change from omg_p/10 to omg_p*10
yp_3 = -1*pi/2*ones(size(omg5));                            % phase is -pi/2 for high frequency
yp(1:length(omg3)) = yp_1;
yp(length(omg3)+1:length(omg3)+length(omg4))= yp_2;
yp(length(omg3)+length(omg4)+1:end)= yp_3;
asym_1_ph = yp;
figure(3),subplot(2,1,1),semilogx(omg,asym_1_ph,'--'),hold on
xlabel('\omega'),ylabel('phase factor')
zp = zeros(size(omg));
zp_1 = pi/2*ones(size(omg3));                               % phase is pi/2 for low frequency
zp_2 = -1*pi/2/2*(log10(omg4)-log10(omg_p*10));
% a -pi/2 phase change from omg_p/10 to omg_p*10
zp_3 = zeros(size(omg5));                                   % phase is 0 for high frequency
zp(1:length(omg3)) = zp_1;
zp(length(omg3)+1:length(omg3)+length(omg4))= zp_2;
zp(length(omg3)+length(omg4)+1:end)= zp_3;
asym_2_ph = zp;
figure(3),subplot(2,1,1),semilogx(omg,asym_2_ph,':'),hold off
asym_m = asym_1+asym_2;                                     % composition for magnitide
figure(2),subplot(2,1,2),semilogx(omg,asym_m),
xlabel('\omega'),ylabel('gain factor')
asym_ph = asym_1_ph+asym_2_ph;                              % composition for phase
figure(3),subplot(2,1,2),semilogx(omg,asym_ph),
xlabel('\omega'),ylabel('phase factor')
```

(c). $H(j\omega) = \dfrac{j20\omega}{10000 - \omega^2 + j20\omega} = \dfrac{20j\omega}{(j\omega)^2 + 20j\omega + 10000}$

Code:

```
omg = 0.01:0.01:10000;
num = [20,0];den = [1,20,10000];
sys = tf(num,den);
figure(1),bode(omg,sys);grid on
```



Asymptote:

$$H(j\omega)=\frac{j20\omega}{(j\omega+10-j99.5)(j\omega+10+j99.5)}$$

$$=\frac{j20\omega}{10000\left[1-\frac{\omega^2}{10000}+j\frac{\omega}{500}\right]}=\frac{j\frac{\omega}{500}}{1-\left(\frac{\omega}{100}\right)^2+j\frac{\omega}{500}}$$

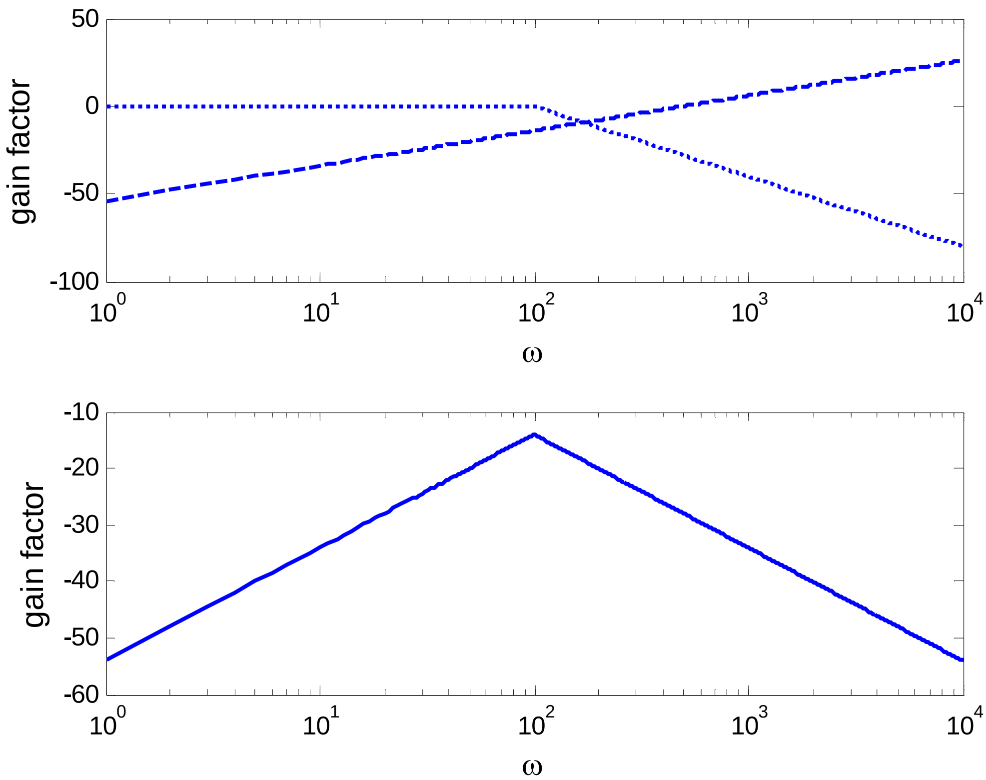
From the expression, we see the frequency response have: zero at $j\omega=0$, a pair of complex conjugate poles at $j\omega=-10+j99.5$ and $j\omega=-10-j99.5$.

(1). Magnitude asymptote

The dB-scale magnitude asymptote from zero $j\frac{\omega}{500}$ is a straight line with a slope of 20dB and go through (500,0), as shown (dashed line).

The dB-scale magnitude asymptote from pole $1-\left(\frac{\omega}{100}\right)^2+j\frac{\omega}{500}$ is a pair of straight lines and $\omega_p=100$. For low frequency ($\omega \ll \omega_p$), it's a straight line with a slope of 0dB. For high frequency ($\omega \gg \omega_p$), it's a straight line with a slope of -40dB. These two asymptotes intersect at $\omega=\omega_p$, as shown (dotted line).

Then, applying composition rules to get the final magnitude asymptote: dashed line plus dotted line, as shown (solid line).

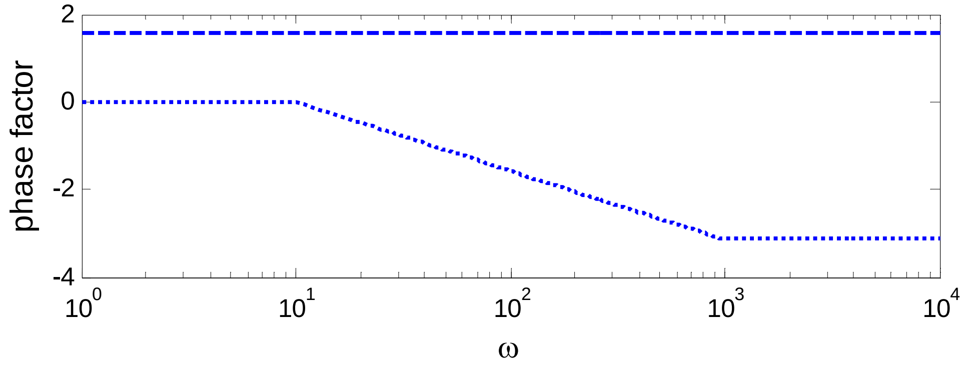


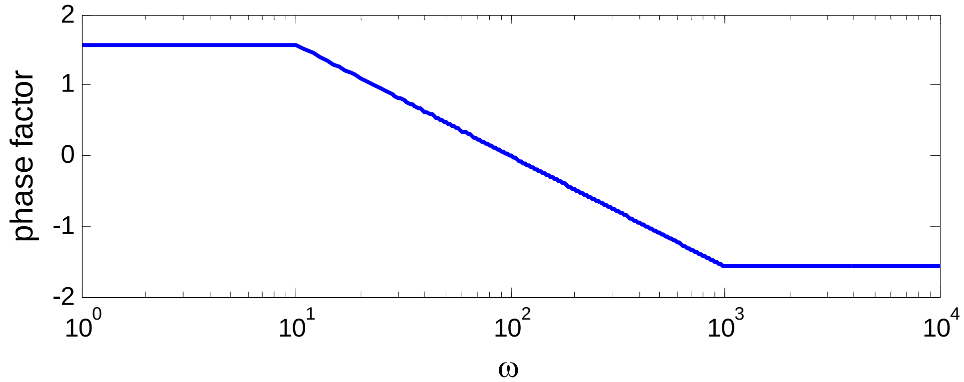
(2). Phase asymptote

The phase asymptote from zero $j\frac{\omega}{500}$ is a straight line with a slope of 0 at phase equals $\frac{\pi}{2}$, as shown (dashed line).

The phase asymptote from pole $1-\left(\frac{\omega}{100}\right)^2+j\frac{\omega}{500}$ contains three segments and

$\omega_p = 100$. For low frequency ($\omega \ll \omega_p/10$), it's a straight line with a slope of 0 at phase equals 0. For high frequency ($\omega \gg 10\omega_p$), it's a straight line with a slope of 0 at phase equals $-\pi$. Then for frequencies between them ($\omega_p/10 < \omega < 10\omega_p$), the straight line (reference to $\log_{10} \omega$) achieves the $-\pi$ phase change, as shown (dotted line). Then, get the phase asymptote by applying composition rules: dashed line plus dotted line, as shown (solid line).





Code:

```
clear all
omg = 1:1:10000;
asym_z = 20*log10(omg/500);                    % get the magnitude asymptote from zero
figure(2),subplot(2,1,1),semilogx(omg,asym_z,'--'),hold on
xlabel('\omega'),ylabel('gain factor')

asym_z_ph = pi/2*ones(size(omg));              % get the phase asymptote from zero
figure(3),subplot(2,1,1),semilogx(omg,asym_z_ph,'--'),hold on
xlabel('\omega'),ylabel('phase factor')

omg_p = 100;
omg1 = 1:1:omg_p;
```

```
omg2 = omg_p+1:1:10000;
ym = zeros(size(omg));              % get the magnitude asymptote from the pole for low frequency
ym_1 = ones(size(omg1));          % get the magnitude asymptote from the pole for high frequency
ym_2 = omg2/omg_p;
ym(1:length(omg1)) = ym_1;
ym(length(omg1)+1:end)= ym_2;
asym_p = -40*log10(ym);
figure(2),subplot(2,1,1),semilogx(omg,asym_p,':'),hold off

omg3 = 1:1:omg_p/10;
omg4 = omg_p/10+1:1:omg_p*10;
omg5 = omg_p*10+1:1:10000;
yp = zeros(size(omg));
yp_1 = zeros(size(omg3));
yp_2 = -1*pi/2*(log10(omg4)-log10(omg_p/10));
% a -pi/2 phase change from omg_p/10 to omg_p*10
yp_3 = -1*pi*ones(size(omg5));
yp(1:length(omg3)) = yp_1;
yp(length(omg3)+1:length(omg3)+length(omg4))= yp_2;
yp(length(omg3)+length(omg4)+1:end)= yp_3;
asym_p_ph = yp;
figure(3),subplot(2,1,1),semilogx(omg,asym_p_ph,':'),hold off

asym_m = asym_z+asym_p;                                   % composition for magnitude asymptote
figure(2),subplot(2,1,2),semilogx(omg,asym_m),
xlabel('\omega'),ylabel('gain factor')
asym_ph = asym_z_ph+asym_p_ph;
figure(3),subplot(2,1,2),semilogx(omg,asym_ph),          % composition for phase asymptote
xlabel('\omega'),ylabel('phase factor')
```