Homework 6

Problem 33. Excitation of the system and its response are given

$$x(t) = 4\,rect(t/2),\ \ y(t) = 10[(1 - e^{-(t+1)})u(t+1) - (1 - e^{-(t-1)})u(t-1)]$$

We know that response is equal to the convolution of excitation and impulse response

$y(t) = x(t) * h(t)$ or in Fourier space $Y(f) = X(f)H(f)$, where convolution in time domain is 'equivalent' to usual multiplication in frequency domain. We can calculate $X(f), Y(f)$. Then $H(f) = Y(f)/X(f)$ and the impulse response $h(t)$ will be the inverse Fourier transform of $H(f)$.

Step#1. $X(f) = 4 \cdot 2 \cdot sinc(2f) = 8sinc(2f)$ (Formulas page 369-371)

$Y(f) = 10F([1 - e^{-t}]u(t)) \cdot [e^{2\pi fj} - e^{-2\pi fj}]$, here F denotes Fourier transform.

$F([1 - e^{-t}]u(t)) = \frac{1}{2}\delta(f) + \frac{1}{2\pi fj} - \frac{1}{1+2\pi fj} = \frac{1}{2}\delta(f) + \frac{1}{2\pi fj} \cdot \frac{1}{1+2\pi fj}$ Then

$$Y(f) = 10\left[\frac{1}{2}\delta(f) + \frac{1}{2\pi fj} \cdot \frac{1}{1+2\pi fj}\right] \cdot 2j\sin(2\pi fj) = \frac{20j\sin(2\pi fj)}{2\pi fj}\frac{1}{1+2\pi fj} = 20sinc(2f) \cdot \frac{1}{1+2\pi fj}$$

$$H(f) = \frac{Y(f)}{X(f)} = 20sinc(2f) \cdot \frac{1}{1+2\pi fj} \cdot \frac{1}{8sinc(2f)} = 2.5\frac{1}{1+2\pi fj}$$

Step#2. We need to find the inverse Fourier transform of $H(f) = 2.5\frac{1}{1+2\pi fj}$. Formulas on pages 369-371 suggest $h(t) = 2.5e^{-t}u(t)$, this is the impulse response of the system.

Problem 34b. $g(t) = 4[\delta_4(t+1) - \delta_4(t-3)]$. One can observe that $\delta_4(t+1) = \delta_4(t-3)$ and $g(t) = 0$ since period of each function is 4. However, let us go to Fourier space and check if we indeed get zero.

$$G(f) = 4F(\delta_4(t)) \cdot [e^{-2\pi fj} - e^{6\pi fj}] = \frac{4}{4}\delta_{\frac{1}{4}}(f)[e^{-2\pi fj} - e^{6\pi fj}] = \delta_{\frac{1}{4}}(f) \cdot [e^{-2\pi fj} - e^{6\pi fj}]$$

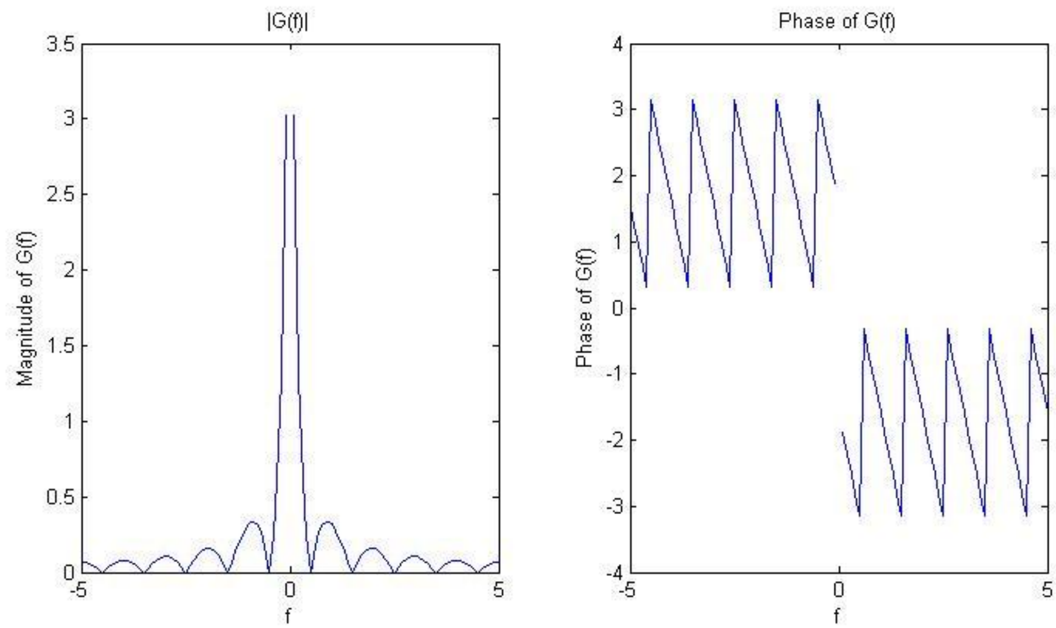
$$= \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{4}\right) \cdot [e^{-2\pi fj} - e^{6\pi fj}] = \sum_{n=-\infty}^{\infty}\left[e^{-\frac{\pi nj}{2}} - e^{\frac{3\pi nj}{2}}\right] = \sum_{n=-\infty}^{\infty} 0 = 0$$

So the magnitude and phase are constant zeros.

Problem 34c. $g(t) = u(2t) + u(t-1)$ First, we notice that $u(2t) = u(t)$ and $g(t) = u(t) + u(t-1)$

$$G(f) = F(u(t)) \cdot [1 + e^{-2\pi f j}] = \left[\frac{1}{2}\delta(f) + \frac{1}{2\pi f j}\right] \cdot [1 + e^{-2\pi f j}] = \delta(f) + \frac{1 + e^{-2\pi f j}}{2\pi f j}$$

Clearly at $f = 0$ the value of $G(f)$ is infinite, one can observe this from the fact that the total area of the signal in the time-domain is infinite.




```
Matlab code:
clear all
close all
clc
f=-5:0.1:5; % frequency range
y=dirac(f)+1/2/i/pi./f.*(1+cos(2*pi*f)-i*sin(2*pi*f)); % Fourier image
subplot(1,2,1)
plot(f,abs(y),'-')
title('|G(f)|')
xlabel('f')
ylabel('Magnitude of G(f)')
subplot(1,2,2)
plot(f,angle(y))
title('Phase of G(f)')
xlabel('f')
ylabel('Phase of G(f)')
```

Problem 39a.

$x_1(t) \leftrightarrow X_1(f)$, $x_2(t) = x_1(t+4)$, then from Formulas on page 369-371

$X_2(f) = X_1(f) \cdot e^{8\pi f j}$. Magnitude of the latter factor is one $|e^{8\pi f j}| = 1$, $|X_1(f)| = |X_2(f)|$

Problem 39b.

$x_1(t) \leftrightarrow X_1(f)$, $x_2(t) = x_1(t/5)$, then from Formulas on page 369-371

$X_2(f) = 5X_1(5f)$. The maximum value of $|X_1(f)|$ is equal to the maximum value of $|X_1(5f)|$.

According to this, $\max|X_2(f)| = 5\max|X_1(f)|$

Problem 39c.

CTFT of $x(t)$ has the value of $e^{-\frac{j\pi}{4}}$ at a frequency $f = 20$. We can assume that signal $x(t)$ is real-valued. Then from the definition of Fourier transform

$$X(20) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi 20t}dt = \left(\int_{-\infty}^{\infty} x(t)e^{j2\pi 20t}dt\right)^* = X^*(-20)$$

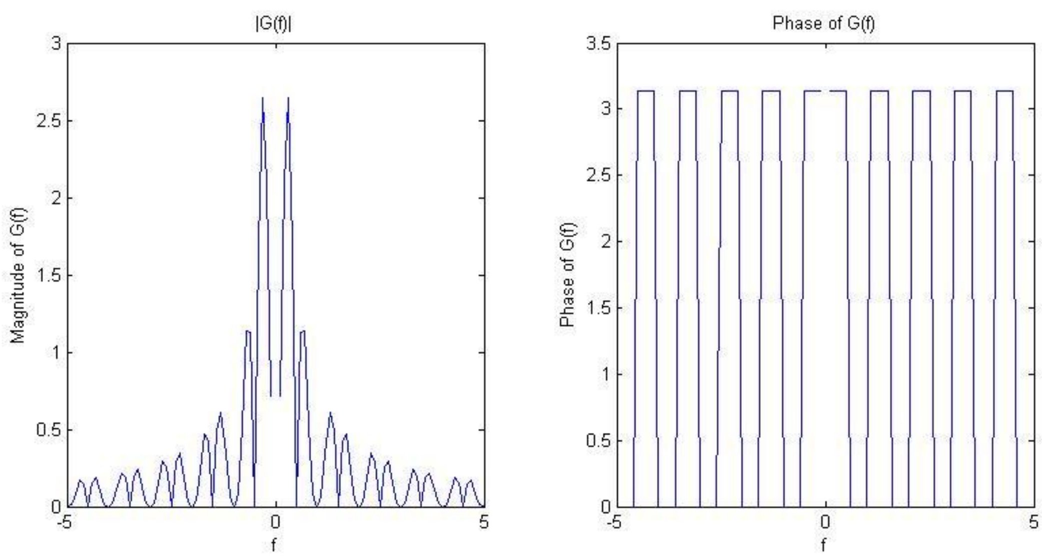
$$X(-20) = X^*(20) = e^{\frac{j\pi}{4}}, \text{ here } (\quad)^* \text{ stands for conjugate.}$$

However, if we do not consider $x(t)$ to be purely real, the answer cannot be given based only on the given data, more information is needed.

Problem 47.

$A = -B = 1,\ \ t_1 = 1,\ \ t_2 = 2$. Then we can express the function $x(t)$ as

$x(t) = rect\left(\frac{t}{4}\right) - 2\,rect\left(\frac{t}{2}\right)$, and then $X(f) = 4sinc(4f) - 4sinc(2f)$.



Matlab code:

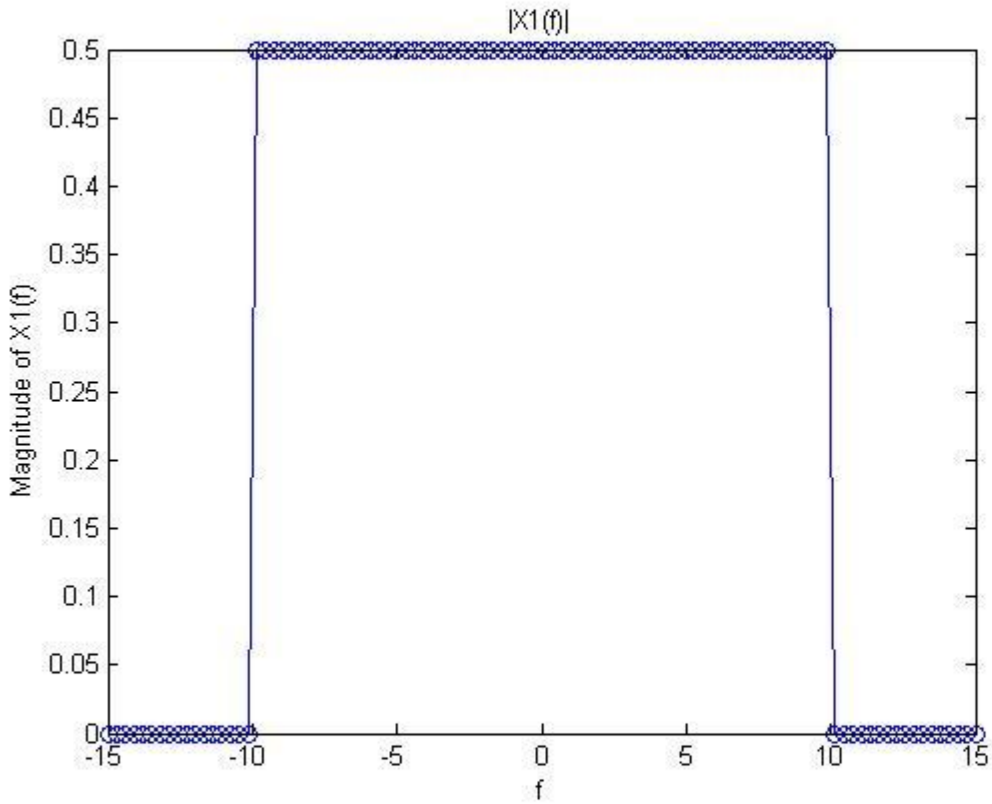
```
clear all
close all
clc
f=-5:0.1:5;
y=4*sin(pi*4*f)./f/pi/4 -4*sin(pi*2*f)./f/pi/2;
subplot(1,2,1)
plot(f,abs(y),'-')
title('|G(f)|')
xlabel('f')
ylabel('Magnitude of G(f)')
subplot(1,2,2)
plot(f,angle(y))
title('Phase of G(f)')
xlabel('f')
```

```
ylabel('Phase of G(f)')
```

Problem 48. $x_1 = 10\,sinc(20t),\ x_2 = 5\cos(2000\pi t),\ y(t) = x_1(t)\cdot x_2(t)$.

Then in Fourier space $Y(f) = X_1(f) * X_2(f)$. Again, using Formulas on pages 369-371

$$X_1(f) = 10\cdot\frac{1}{20}\cdot rect\left(\frac{f}{20}\right),\quad X_2(f) = 5\cdot\frac{1}{2}\cdot[\delta(f-1000)+\delta(f+1000)].$$



Matlab code:

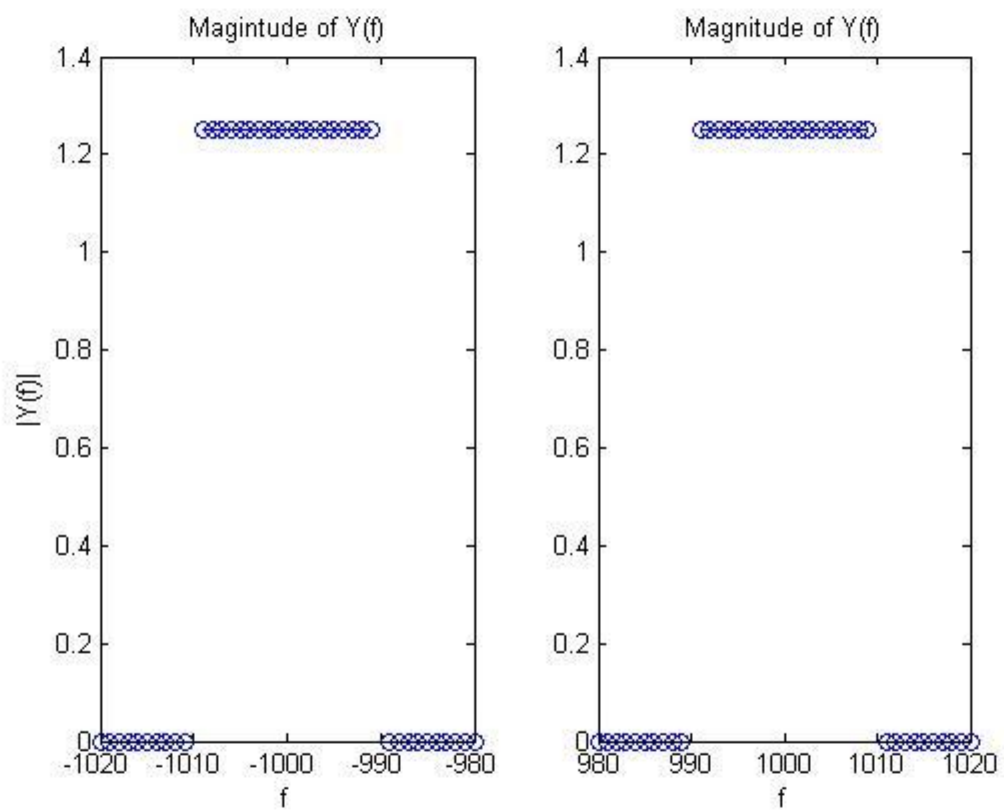
```
% defining function that computes rect(t)
function y = rect(t)
y=heaviside (t+0.5)-heaviside(t-0.5);
end

clear all
close all
clc

f1=linspace(-15,15);
x1=1/2*rect(f1/20);
plot(f1,abs(x1),'-o')
title('|X1(f)|')
xlabel('f')
ylabel('Magnitude of X1(f)')
```

$Y(f) = \int_{-\infty}^{\infty} \frac{1}{2} rect\left(\frac{f-s}{20}\right) \cdot \frac{5}{2} \cdot [\delta(s-1000) + \delta(s+1000)]ds = \frac{5}{4}[rect\left(\frac{f-1000}{20}\right) + rect(\frac{f+1000}{20})]$

Locations of the rect functions are centered at $f = -1000, f = 1000$ with height 1.25.



Matlab code:

```
function y = rect(t)
y=heaviside (t+0.5)-heaviside(t-0.5);
end

clear all
close all
clc

f1=[-1020:1:-980]; % location of the first rectangular
y1=5/4*(rect(f1/20-50)+rect(f1/20+50));
subplot(1,2,1)
plot(f1,abs(y1),'-o')
title('Magintude of Y(f)')
xlabel('f')
ylabel('|Y(f)|')
f2=[980:1020]; % location of the second rectangular
y2=5/4*(rect(f2/20-50)+rect(f2/20+50));
subplot(1,2,2)
```

```
plot(f2,abs(y2),'-o')
title('Magnitude of Y(f)')
xlabel('f')
```

In simple words, mixer is producing the signal with frequencies equal to the sum and difference of the corresponding frequencies of 2 input signals.

Problem 49. $h(t) = A[u(t) - u(t - t_0)]$. We want to reject 60Hz and all its harmonics. Let us look at the Fourier transform of the impulse response

$$H(f) = AF\big(u(t)\big) \cdot \left[1 - e^{-2\pi f j t_0}\right] = A\left(\frac{1}{2}\delta(f) + \frac{1}{2\pi f j}\right) \cdot \left[1 - e^{-2\pi f j t_0}\right] = \frac{A}{2\pi f j}\left[1 - e^{-2\pi f j t_0}\right]$$

We need $H(f) = 0,\ f = 60, 120, 180, \ldots$ This is possible when $\left[1 - e^{-2\pi f j t_0}\right] = 0$, or in other words $2\pi f t_0 = 2\pi n$ where n is integer. Letting $t_0 = \frac{1}{60}$ we will exclude all desired harmonics.