Homework 7 Solutions

Problem 25: $x(t) = rect(1000t) * \delta_{0.002}(t)$ where $w = 1/1000, T_0 = 0.002$, and $f_0 = \frac{1}{T_0} = 500$

Fourier transform the input signal and multiple with the frequency response of the ideal filter. The inverse of the product yields the output signal.

$$y(t) = h(t)x(t) \xrightarrow{\mathcal{F}} Y(f) = H(f) * X(f)$$

First, recall the CTFS properties of multiplication – convolution duality and transforms of periodic signal.

$$x(t) = a(t) * b(t) \xrightarrow{\mathcal{F}} X(f) = A(f)B(f)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{-j2\pi k f_s t} \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} X[k]\,\delta\,(f - kf_s)$$

Therefore, the Fourier transform of the input signal is

$$x(t) = rect\left(\frac{t}{w}\right) * \delta_{T_0}(t) \xrightarrow{\mathcal{F}} X(f) = wsinc(wf)f_0\delta_{f_0}(f) = wf_0 \sum_{k=-\infty}^{\infty} sinc(wkf_0)\delta\,(f - kf_0)$$
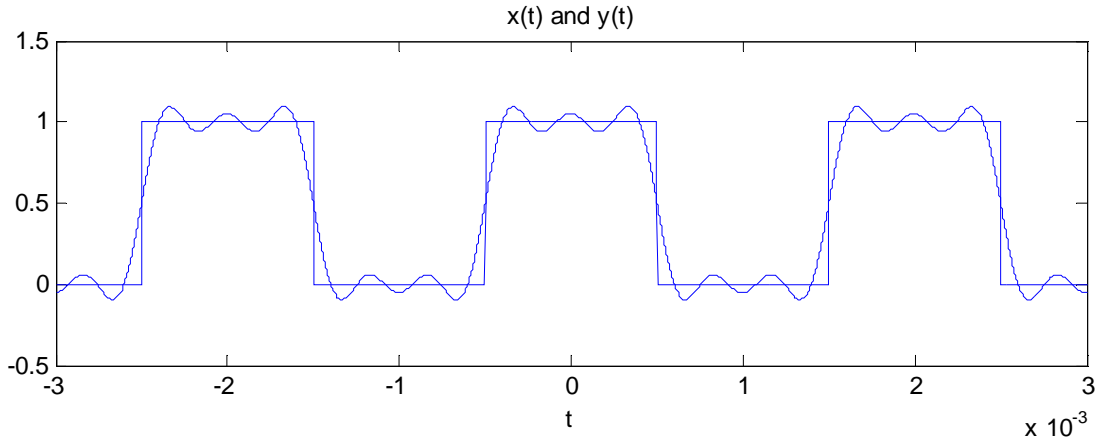
$$X(f) = \frac{1}{2} \sum_{k=-\infty}^{\infty} sinc\left(\frac{k}{2}\right)\delta\,(f - 50k)$$

a) The frequency response of an ideal lowpass filter with a cutoff frequency of 3 kHz is

$$H(f) = rect\left(\frac{f}{2f_m}\right) = rect\left(\frac{f}{6000}\right)$$

$$Y(f) = H(f) * X(f) = rect\left(\frac{f}{6000}\right) * \frac{1}{2} \sum_{k=-\infty}^{\infty} sinc\left(\frac{k}{2}\right)\delta\,(f - 50k)$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} rect\left(\frac{k}{12}\right)sinc\left(\frac{k}{2}\right)\delta\,(f - 50k) = \frac{1}{2} \sum_{k=-6}^{6} sinc\left(\frac{k}{2}\right)\delta\,(f - 50k)$$

$$= \frac{1}{2}\left[sinc\left(\frac{5}{2}\right)[\delta(f - 2500) + \delta(f + 2500)]\right.$$

$$+ sinc\left(\frac{3}{2}\right)[\delta(f - 1500) + \delta(f + 1500)]$$

$$+ sinc\left(\frac{1}{2}\right)[\delta(f - 500) + \delta(f + 500)] + \left.\delta(f)\right]$$

$$y(t) = \frac{1}{2}\left[2sinc\left(\frac{5}{2}\right)\cos(5000\pi t) + 2sinc\left(\frac{3}{2}\right)\cos(3000\pi t) + 2sinc\left(\frac{1}{2}\right)\cos(1000\pi t) + 1\right]$$

x(t) and y(t)


b) The frequency response of an ideal bandpass filter with a low cutoff frequency of 1 kHz and a high cutoff frequency of 5 kHz is

$$H(f) = rect((f - f_0)/\Delta f) + rect((f + f_0)/\Delta f) = rect\left(\frac{f - 3000}{4000}\right) + rect\left(\frac{f + 3000}{4000}\right)$$

$$where\ f_0 = \frac{(f_H + f_L)}{2} = 3000\ and\ \Delta f = f_H - f_L = 4000$$

$$Y(f) = H(f)X(f) = \frac{1}{2}\left[rect\left(\frac{f - 3000}{4000}\right) + rect\left(\frac{f + 3000}{4000}\right)\right] \sum_{k=-\infty}^{\infty} sinc\left(\frac{k}{2}\right) \delta\ (f - 500k)$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} \left[rect\left(\frac{k}{80} - \frac{3}{4}\right) + rect\left(\frac{k}{80} + \frac{3}{4}\right)\right] sinc\left(\frac{k}{2}\right) \delta\ (f - 500k)$$

$$= \frac{1}{2}\left[\sum_{k=-10}^{-2} sinc\left(\frac{k}{2}\right) \delta\ (f - 500k) + \sum_{k=10}^{2} sinc\left(\frac{k}{2}\right) \delta\ (f - 500k)\right]$$

$$= \frac{1}{2}\left[sinc\left(\frac{9}{2}\right)[\delta(f - 4500) + \delta(f + 4500)]\right.$$

$$+ sinc\left(\frac{7}{2}\right)[\delta(f - 3500) + \delta(f + 3500)] + sinc\left(\frac{5}{2}\right)[\delta(f - 2500) + \delta(f + 2500)]$$

$$+ sinc\left(\frac{3}{2}\right)[\delta(f - 1500) + \delta(f + 1500)]\right]$$

$$y(t) = sinc\left(\frac{9}{2}\right)\cos(9000\pi t) + sinc\left(\frac{7}{2}\right)\cos(7000\pi t) + sinc\left(\frac{5}{2}\right)\cos(5000\pi t)$$

$$+ sinc\left(\frac{3}{2}\right)\cos(3000\pi t)$$
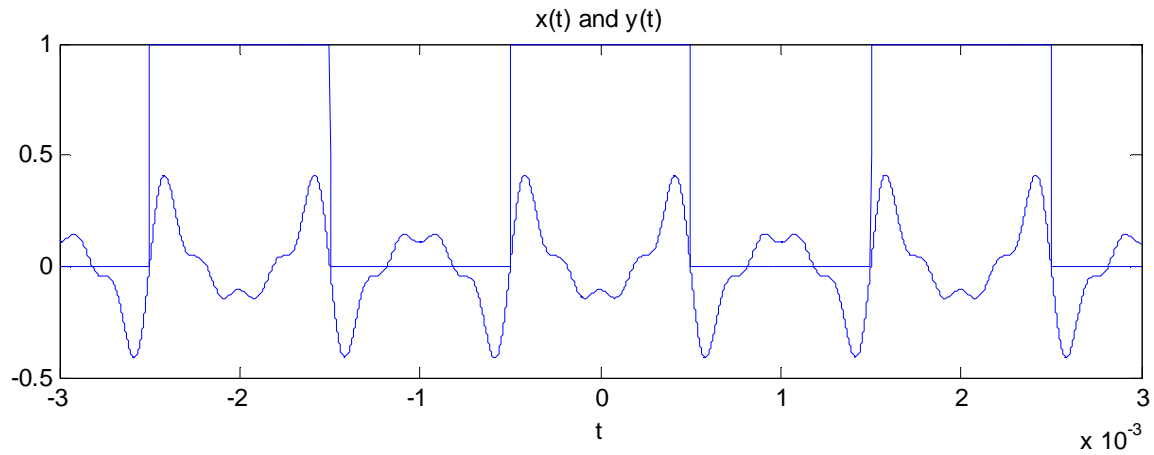
x(t) and y(t)

Matlab Code:

```matlab
% Part a
close all;clear all;clc

t = -0.003:0.000001:0.003;
y = zeros(size(t));
x = zeros(size(t));

for k = 1:6
    y = y + 1/2*[sinc(k/2)*2*cos(1000*pi*t*k)];
end
y = y + 1/2;

for n = -2:2
    x = x + rect(1000*(t-0.002*n));
end

plot(t,y)
hold on
plot(t,x); title('x(t) and y(t)'); xlabel('t');

% Part b
t = -0.003:0.000001:0.003;
y1 = zeros(size(t));
y2 = zeros(size(t));
x = zeros(size(t));

for k = 2:10
    y1 = y1 + sinc(k/2)*cos(1000*pi*t*k);
end

for k = -2:-10
    y2 = y2 + sinc(k/2)*cos(1000*pi*t*k);
end

y = y1 + y2;

for k = -2:2
```

```
    x = x + rect(1000*(t-0.002*k));
end

plot(t,y)
hold on
plot(t,x); title('x(t) and y(t)'); xlabel('t');
```

Problem 31:

a)   This is an ideal lowpass filter because its impulse response is a sinc function with an arbitrary time delay. Thus its frequency response is a rectangular function. For t < 0, the impulse response is not zero therefore it is non-causal.

b)   This is a practical RLC bandpass filter because the overall behavior of the circuit passes frequencies near the resonant frequency and stops other frequencies. The capacitor passes signals at very low frequencies and stops signals at very high frequencies. The inductor passes signals at very high frequencies and stops signals at very low frequencies. At very low and very high frequencies, the output voltage is practical zero. At the resonant frequency, the impedance goes to infinity therefore the output voltage signal is the same as the input voltage signal. The impulse response is zero for t < 0 therefore it is causal.

c)   This is a practical lowpass filter because the magnitude of the frequency response is greatest at very low frequencies. At high frequencies, the signal is stopped as the magnitude of the frequency response goes to zero. Whether it is causal or non-casual cannot be determined without the phase plot.

Problem 40:   From the Bode diagram, the following can be concluded:

*   Gain of 6 dB from -30 dB to -24 dB
*   One real zero at 6 kHz and one real pole at 12 kHz

This specifies an active highpass filter. The Body diagram of this filter must have one real zero at 6 kHz and one real pole at 12 kHz. Therefore the transfer function must take the general form of $H(s) = (s + a)/(s + b)$ where a is the one real zero and b is the one real pole.

There are two circuits that can satisfy this problem. First circuit is one with two capacitors and two resistors in series.
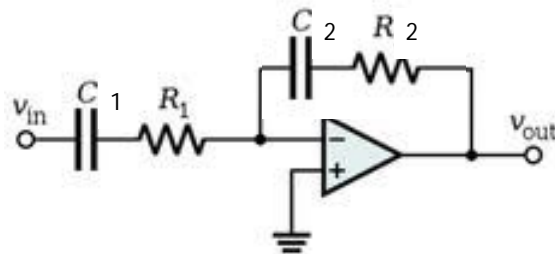
The transfer function is then

$$H(s) = \frac{R_2 + \frac{1}{sC_2}}{R_1 + \frac{1}{sC_1}} = \frac{R_2}{R_1} \frac{s + \frac{1}{R_2C_2}}{s + \frac{1}{R_1C_2}}$$

where $\frac{R_2}{R_1} = 1, R_1C_1 = \frac{1}{b}, R_2C_2 = \frac{1}{a}$

The second filter can be calculated through the highpass filters in Figure 12.57. Filter (a) has finite gain at very low frequencies and a gain that rises with frequency at high frequencies, never approaching a constant. Thus it has one real zero. Filter(b) has a gain that falls with frequency at low frequencies, approaching zero at zero frequency, and approaches a constant gain at high frequencies. Thus it has one real pole.

In our problem, there is a finite gain at very low frequencies and a gain of 6 dB that rises between 6 kHz and 12 kHz, and then approaches a constant gain. A combination of both filter (a) and (b) would satisfy our requirements which yields the active highpass filter below.
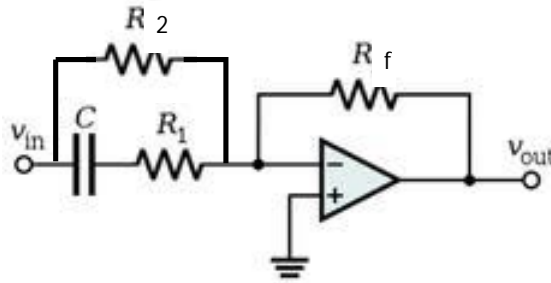
For this filter, the overall gain is

$$H(f) = -\frac{Z_f}{Z_i} = -\frac{R_f}{\frac{R_2\left(R_1 + \frac{1}{j2\pi fC}\right)}{R_1 + R_2 + \frac{1}{j2\pi fC}}} = -R_f \frac{R_1 + R_2 + \frac{1}{j2\pi fC}}{R_2\left(R_1 + \frac{1}{j2\pi fC}\right)} = -R_f \frac{j2\pi fC(R_1 + R_2) + 1}{j2\pi fCR_1R_2 + R_2} = -R_f \frac{R_1 + R_2}{R_1R_2} \frac{j2\pi f + \frac{1}{C(R_1+R_2)}}{j2\pi f + \frac{1}{CR_1}}$$

The numerator solves for the one real zero and the denominator for the one real pole.

Zero at 6 kHz $\quad j2\pi f + \frac{1}{C(R_1+R_2)} = 0 \qquad \frac{1}{C(R_1+R_2)} = 2\pi f = 2\pi * 6000\ Hz$

Pole at 12 kHz $\quad j2\pi f + \frac{1}{CR_1} = 0 \qquad \frac{1}{CR_1} = 2\pi f = 2\pi * 12000\ Hz$

$$R_1 = R_2 = \frac{1}{2400\pi C}$$

At low frequencies, the impedance of the capacitor is much greater than the impedance of the resistor. The impedance of the capacitor can then be discounted when you calculate the gain, therefore

$$H(f) = -\frac{R_f}{R_2}$$

Problem 42: $x_t(t) = sinc(t)$, $f_c = 10$, and cutoff frequency of lowpass filter is 1 Hz

The system passes a signal through a modulated carrier, ideal bandpass filter, modulated carrier, and low pass filter, respectively.

$$x_t(t) = sinc(t) \xleftrightarrow{\mathcal{F}} X_t(f) = rect(f)$$
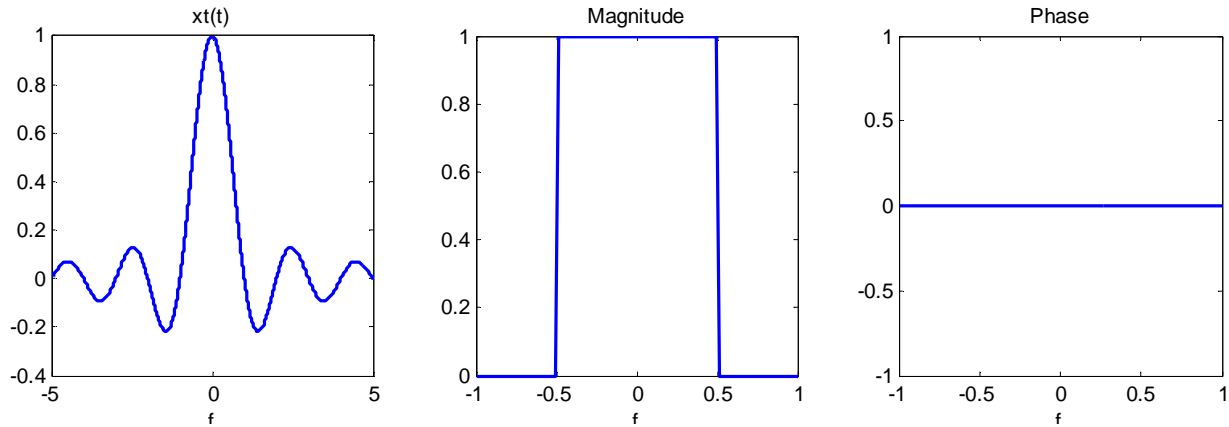
$y_t(t)$ is the product of the input signal, modulated carrier, and ideal bandpass filter.

$$y_t(t) = x_t(t)\cos(2\pi f_c t)h(t)$$

$$Y_t(f) = \frac{1}{2}[X_t(f - f_c) + X_t(f + f_c)]H(f)$$

The ideal bandpass filter has a frequency response of

$$H(f) = rect((f - f_0)/\Delta f) + rect((f + f_0)/\Delta f) = rect\left(\frac{f - 10.25}{0.5}\right) + rect\left(\frac{f + 10.25}{0.5}\right)$$

$$where \ f_0 = \frac{(f_H + f_L)}{2} = \frac{(f_c + f_m) + f_c}{2} = f_c + \frac{f_m}{2} = 10 + \frac{0.5}{2} = 10.25 \ and \ \Delta f = f_H - f_L = f_m = 0.5$$

$$Y_t(f) = \frac{1}{2}[rect(f - 10) + rect(f + 10)]\left[rect\left(\frac{f - 10.25}{0.5}\right) + rect\left(\frac{f + 10.25}{0.5}\right)\right]$$

$$= \frac{1}{2}\left[rect(f - 10)rect\left(\frac{f - 10.25}{0.5}\right) + rect(f - 10)rect\left(\frac{f + 10.25}{0.5}\right)\right.$$

$$+ rect(f + 10)rect\left(\frac{f - 10.25}{0.5}\right) + rect(f + 10)rect\left.\left(\frac{f + 10.25}{0.5}\right)\right]$$

$$= \frac{1}{2}\left[rect(f - 10)rect\left(\frac{f - 10.25}{0.5}\right) + rect(f + 10)rect\left(\frac{f + 10.25}{0.5}\right)\right]$$

As can be seen from the graph,

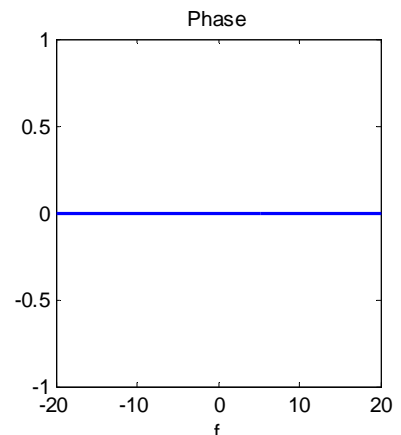$$rect(f - 10)\,rect\left(\frac{f - 10.25}{0.5}\right) = rect\left(\frac{f - 10.25}{0.5}\right)$$
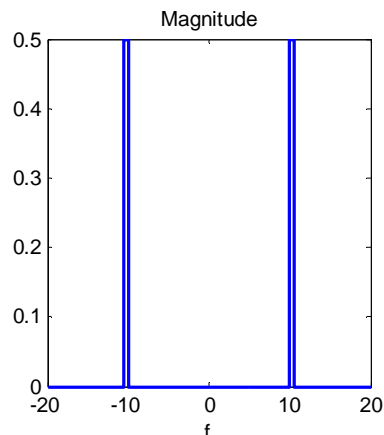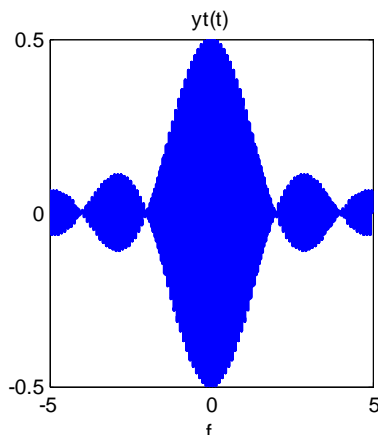
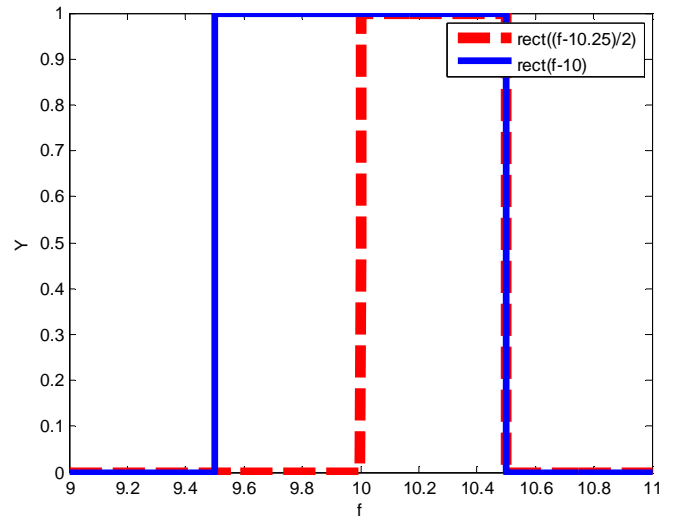$$rect(f + 10)\,rect\left(\frac{f + 10.25}{0.5}\right) = rect\left(\frac{f + 10.25}{0.5}\right)$$

Therefore,

$$Y_t(f) = \frac{1}{2}\left[rect\left(\frac{f - 10.25}{0.5}\right) + rect\left(\frac{f + 10.25}{0.5}\right)\right]$$

$$y_t(t) = \frac{1}{2}\left[\frac{1}{2}\,sinc\left(\frac{t}{2}\right)e^{+j2\pi(10.25)t}\right.$$
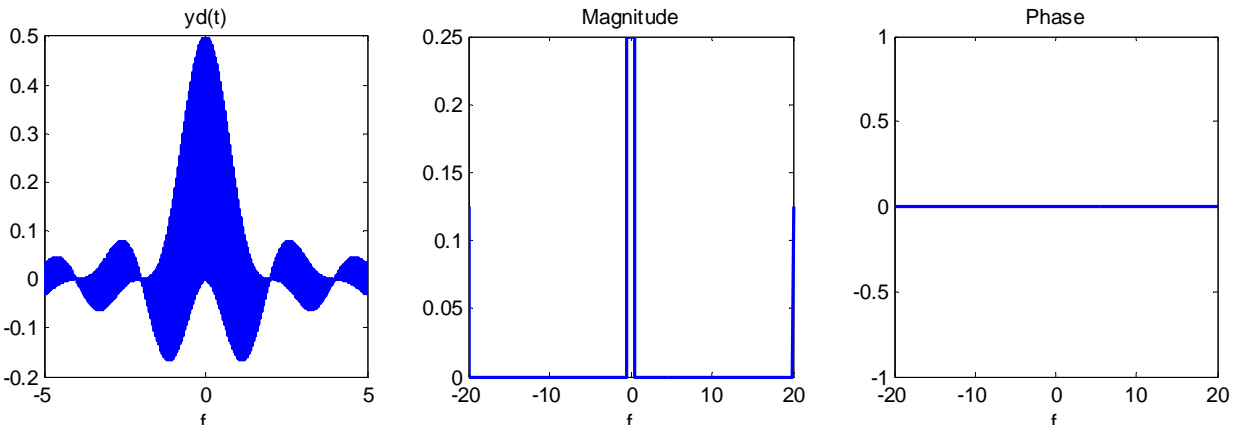$$\left.+ \frac{1}{2}\,sinc\left(\frac{t}{2}\right)e^{-j2\pi(10.25)t}\right]$$

$$y_t(t) = \frac{1}{2}\,sinc\left(\frac{t}{2}\right)cos(20.5\pi t)$$

$y_d(t)$ is a product of $y_t(t)$ and a modulated carrier.

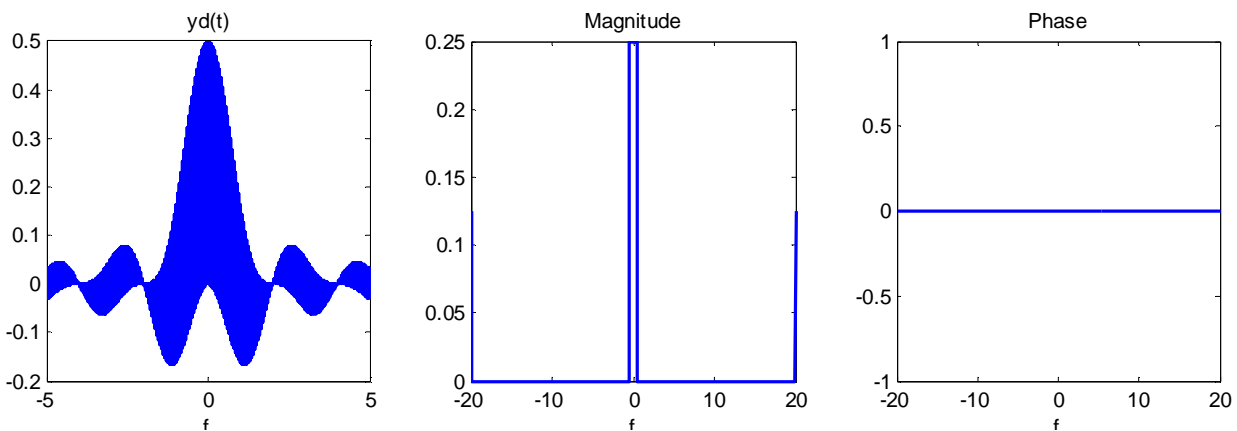$$y_d(t) = y_t(t)cos(20\pi t) = \frac{1}{2}\,sinc\left(\frac{t}{2}\right)cos(20.5\pi t)cos(20\pi t)$$

$$Y_d(f) = \frac{1}{2}[2rect(2f)] * \frac{1}{2}[\delta(f - 10.25) - \delta(f + 10.25)] * \frac{1}{2}[\delta(f - 10) - \delta(f + 10)]$$
$$= \frac{1}{4}rect(2f)[\delta(f - 20.25) + \delta(f - 0.25) + \delta(f + 0.25) + \delta(f + 20.25)]$$
$$= \frac{1}{4}[rect(2(f - 20.25)) + rect(2(f - 0.25)) + rect(2(f + 0.25))$$
$$+ rect(2(f + 20.25))]$$

The lowpass filter stops signal power above the cutoff frequency of 1 Hz which yields

$$Y_f(f) = \frac{1}{4}\left[rect\big(2(f - 0.25)\big) + rect\big(2(f + 0.25)\big)\right] = \frac{1}{4}rect(f)$$

$$y_f(t) = \frac{1}{4}sinc(t)$$

Matlab Code:

```
clear all;close all;clc

t = -5:0.0001:5;
f1 = -1:0.01:1;
f= -20:0.01:20;

xt = sinc(t);
yt = 1/2.*sinc(t./2).*cos(20.5*pi.*t);
yd = 1/2.*sinc(t./2).*cos(20.5*pi.*t).*cos(20*pi.*t);
yf = 1/4.*sinc(t);

Xt = rect(f1);
Yt = 1/2.*[rect((f-10.25)./0.5)+rect((f+10.25)./0.5)];
Yd = 1/4.*[rect(2.*(f-20.25))+rect(2.*(f-0.25))+rect(2.*(f+0.25))+rect(2.*(f+20.25))];
Yf = 1/4.*rect(f1);

figure(1);
subplot(1,3,1); plot(t,xt,'linewidth',2); title('xt(t)'); xlabel('f')
```

```
subplot(1,3,2); plot(f1,abs(Xt),'linewidth',2); title('Magnitude'); xlabel('f')
subplot(1,3,3); plot(f1,angle(Xt),'linewidth',2); title('Phase'); xlabel('f')

figure(2);
subplot(1,3,1); plot(t,yt,'linewidth',2); title('yt(t)'); xlabel('f')
subplot(1,3,2); plot(f,abs(Yt),'linewidth',2); title('Magnitude'); xlabel('f')
subplot(1,3,3); plot(f,angle(Yt),'linewidth',2); title('Phase'); xlabel('f')

figure(3);
subplot(1,3,1); plot(t,yd,'linewidth',2); title('yd(t)'); xlabel('f')
subplot(1,3,2); plot(f,abs(Yd),'linewidth',2); title('Magnitude'); xlabel('f')
subplot(1,3,3); plot(f,angle(Yd),'linewidth',2); title('Phase'); xlabel('f')

figure(4);
subplot(1,3,1); plot(t,yf,'linewidth',2); title('yf(t)'); xlabel('f')
subplot(1,3,2); plot(f1,abs(Yf),'linewidth',2); title('Magnitude'); xlabel('f')
subplot(1,3,3); plot(f1,angle(Yf),'linewidth',2); title('Phase'); xlabel('f')
```

Problem 47: Pulse train $p(t) = (1/w)rect(t/w) * \delta_{\frac{1}{4}}(t)$ is modulated by $x(t) = sinc(t)$

The output signal is $y(t) = x(t)p(t) = sinc(t)\left[(1/\omega)rect(t/\omega) * \delta_{\frac{1}{4}}(t)\right]$

Solve for the CTFS of the output signal using the CTFS properties below.

$$y(t) = x(t)p(t) \xrightarrow{\mathcal{F}} Y(f) = X(f) * P(f)$$

$$p(t) = a(t) * b(t) \xrightarrow{\mathcal{F}} P(f) = A(f)B(f)$$

$$p(t) = \sum_{k=-\infty}^{\infty} P[k]e^{-j2\pi k f_s t} \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} P[k]\,\delta\,(f - kf_s)$$

Therefore, the Fourier transforms are

$$x(t) = sinc(t) \xrightarrow{\mathcal{F}} X(f) = rect(f)$$

$$p(t) = \frac{1}{w}rect\left(\frac{t}{w}\right) * \delta_{T_s}(t) \xrightarrow{\mathcal{F}} P(f) = sinc(wf)f_s\delta_{f_s}(f) = f_s\sum_{k=-\infty}^{\infty} sinc(wkf_s)\delta\,(f - kf_s)$$

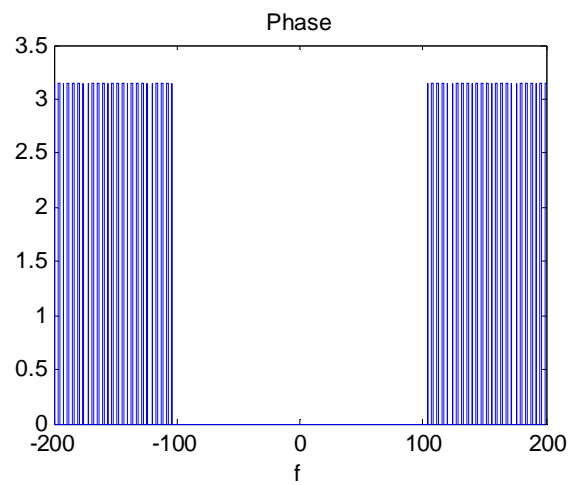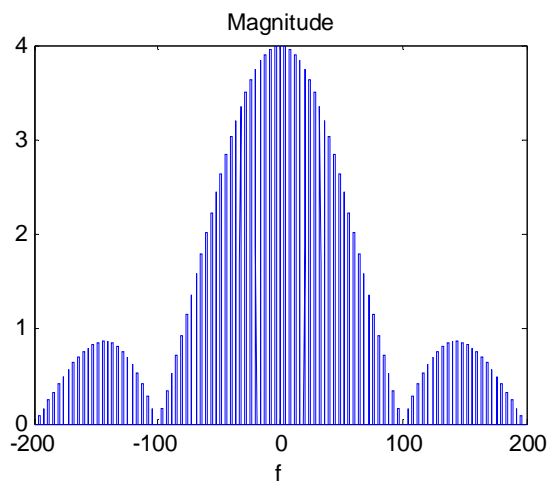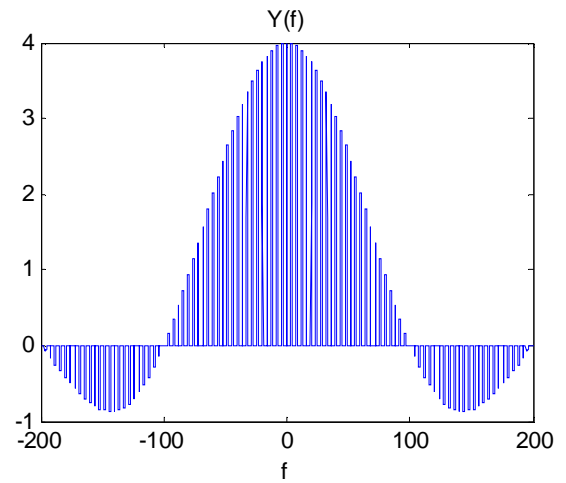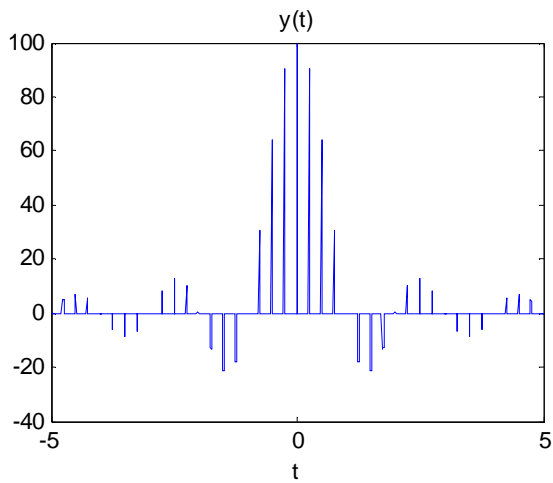$$P(f) = 4\sum_{k=-\infty}^{\infty} sinc(4wk)\delta\,(f - 50k)$$

The convolution of the two yield the CTFS of the output signal

$$Y(f) = X(f) * P(f) = X(f) * \left[f_s\sum_{k=-\infty}^{\infty} sinc(wkf_s)\delta\,(f - kf_s)\right] = f_s\sum_{k=-\infty}^{\infty} sinc(wkf_s)X\,(f - kf_s)$$

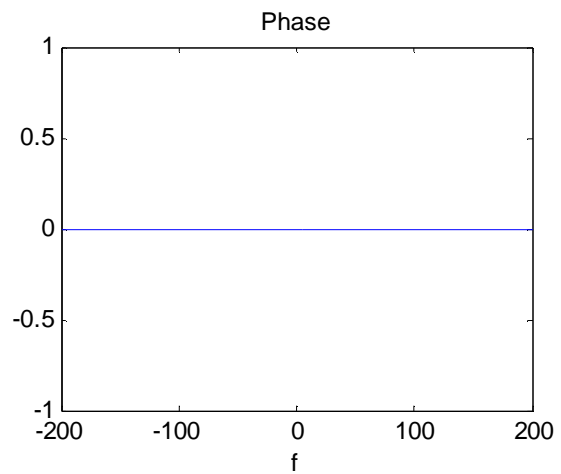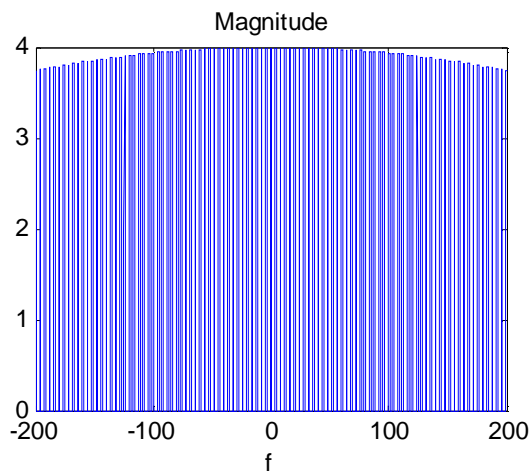$$Y(f) = 4 \sum_{k=-\infty}^{\infty} sinc(4wk)rect(f - 4k)$$

a) $w = 10 \ ms$
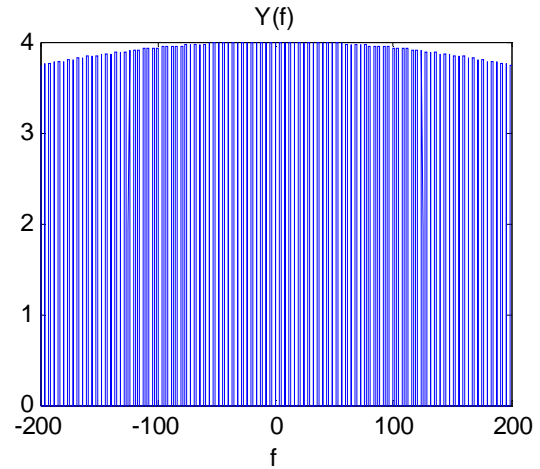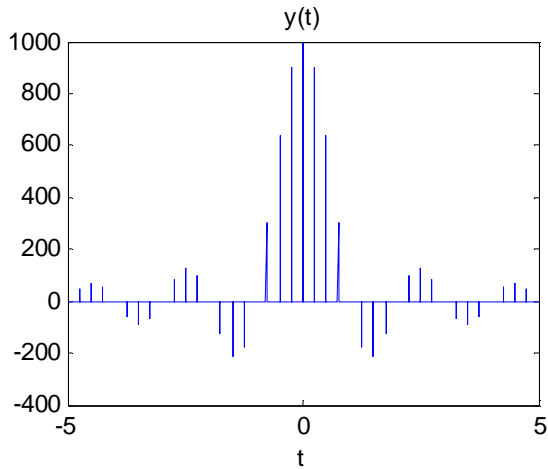
$$Y(f) = 4 \sum_{k=-\infty}^{\infty} sinc(0.04k)rect(f - 4k)$$

b) $w = 1 \ ms$

$$Y(f) = 4 \sum_{k=-\infty}^{\infty} sinc(0.004k)rect(f - 4k)$$

Matlab Code:

```
% Part a
close all; clear all; clc

t = -5:0.01:5;
f = -200:0.01:200;
y = zeros(size(t));
Y = zeros(size(f));

for k = -200:200
    y = y + 100*rect(100*(t-.25*k));
    Y = Y + 4*sinc(0.04*k)*rect(f-4*k);
end

ynew = y.*sinc(t);
Ynew = Y;

subplot(2,2,1); plot(t,ynew); title('y(t)'); xlabel('t');
subplot(2,2,2); plot(f,Ynew); title('Y(f)'); xlabel('f');
subplot(2,2,3); plot(f,abs(Ynew)); title('Magnitude'); xlabel('f');
subplot(2,2,4); plot(f,angle(Ynew)); title('Phase'); xlabel('f');
```

```matlab
% Part b
close all; clear all; clc

t = -5:1e-4:5;
f = -200:0.01:200;
y = zeros(size(t));
Y = zeros(size(f));

for k = -200:200
    y = y + 1000*rect(1000*(t-.25*k));
    Y = Y + 4*sinc(0.004*k)*rect(f-4*k);
end

ynew = y.*sinc(t);
Ynew = Y;

subplot(2,2,1); plot(t,ynew); title('y(t)'); xlabel('t');
subplot(2,2,2); plot(f,Ynew); title('Y(f)'); xlabel('f');
subplot(2,2,3); plot(f,abs(Ynew)); title('Magnitude'); xlabel('f');
subplot(2,2,4); plot(f,angle(Ynew)); title('Phase'); xlabel('f');
```