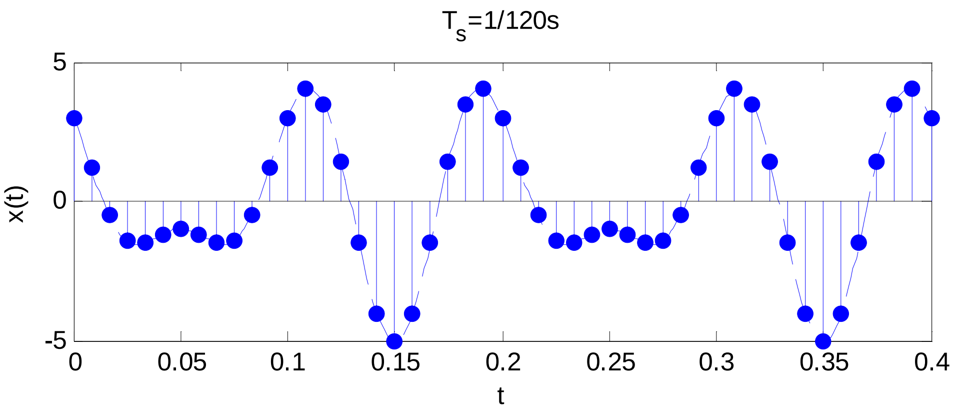
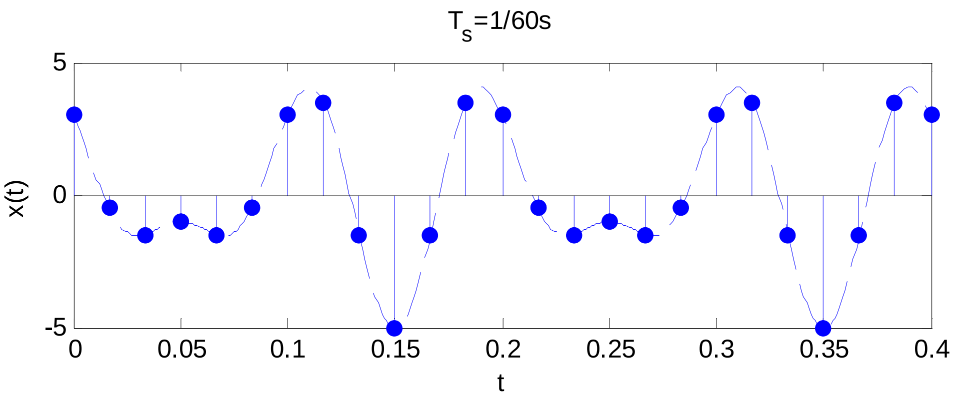
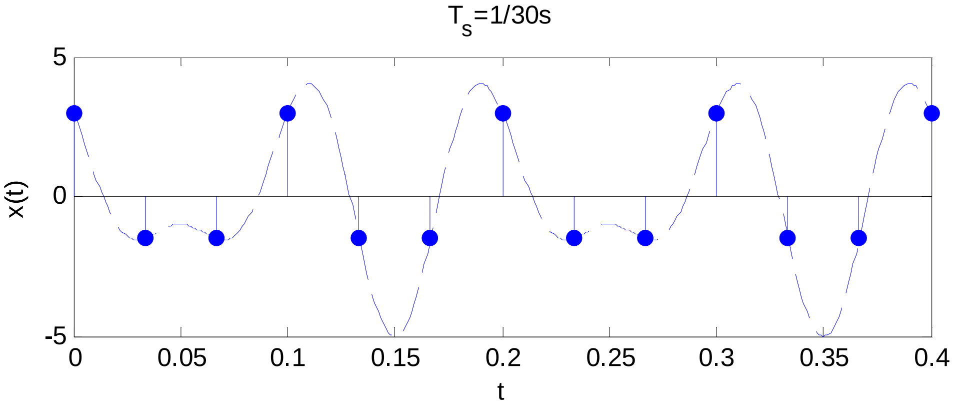
Homework 8 Solutions

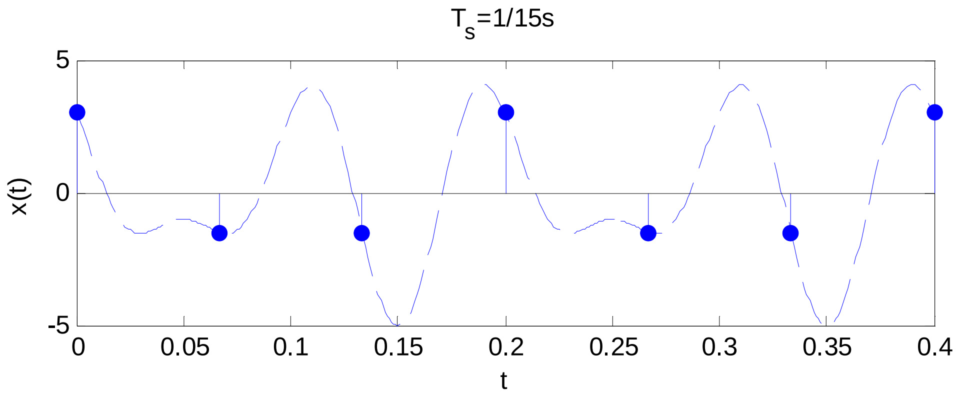
Chapter 14

25. Over a time range of $0 < t < 400ms$, signal $x(t) = 3\cos(20\pi t) - 2\sin(30\pi t)$ is shown in following figures (dashed line), together with sampled by different sampling intervals: 1/120s, 1/60s, 1/30s, 1/15s.









From four figures shown above, this signal can be reconstructed when sampled by $T_s = 1/120s$, $T_s = 1/60s$ and cannot be reconstructed for $T_s = 1/30s$, $T_s = 1/15s$.

Analytically, we can determine if the signal can be reconstructed by finding its Nyquist rate.

$$x(t) = 3\cos(20\pi t) - 2\sin(30\pi t)$$

$$\leftrightarrow X[f] = \frac{3}{2}[\delta(f-10)+\delta(f+10)] + j[\delta(f-15)-\delta(f+15)]$$

So, $f_m = 15\text{Hz}$, $f_{Nyq} = 2f_m = 30\text{Hz}$. In order to reconstruct the signal, sampling frequency should satisfy: $f_s > f_{Nyq} = 30\text{Hz} \Rightarrow T_s < 1/30s$

CODE:

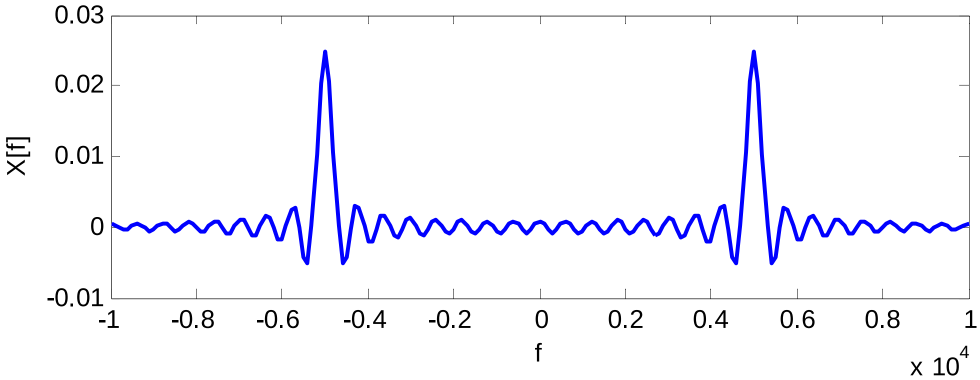
```
clear all;
t = 0:1e-3:400e-3;
y0 = 3*cos(20*pi*t)-2*sin(30*pi*t);
figure(1),
subplot(2,1,1),plot(t,y0,'--');
xlabel('t');ylabel('x(t)'),hold on
subplot(2,1,2),plot(t,y0,'--');
xlabel('t');ylabel('x(t)'),hold on
figure(2),
subplot(2,1,1),plot(t,y0,'--');
xlabel('t');ylabel('x(t)'),hold on
subplot(2,1,2),plot(t,y0,'--');
xlabel('t');ylabel('x(t)'),hold on
t1 = 0:1/120:400e-3;                % (a) Ts = 1/120s;
y1 = 3*cos(20*pi*t1)-2*sin(30*pi*t1);
figure(1)
subplot(2,1,1),stem(t1,y1,'fill');
title('T_s=1/120s'),hold off
t2 = 0:1/60:400e-3;                 % (b) Ts = 1/60s;
```

```
y2 = 3*cos(20*pi*t2)-2*sin(30*pi*t2);
figure(1)
subplot(2,1,2),stem(t2,y2,'fill');
title('T_s=1/60s'),hold off
t3 = 0:1/30:400e-3;                    % (c) Ts = 1/30s;
y3 = 3*cos(20*pi*t3)-2*sin(30*pi*t3);
figure(2)
subplot(2,1,1),stem(t3,y3,'fill');
title('T_s=1/30s'),hold off
t4 = 0:1/15:400e-3;                    % (d) Ts = 1/15s;
y4 = 3*cos(20*pi*t4)-2*sin(30*pi*t4);
figure(2)
subplot(2,1,2),stem(t4,y4,'fill');
title('T_s=1/15s'),hold off
```

32. (a) $x(t) = 15\text{rect}(300t)\cos(10^4\pi t)$

$$X[f] = \frac{15}{300}\text{sinc}(\frac{f}{300}) * \frac{1}{2}[\delta(f-5000)+\delta(f+5000)]$$

$$= \frac{1}{40}\left[\text{sinc}\left(\frac{f-5000}{300}\right)+\text{sinc}\left(\frac{f+5000}{300}\right)\right]$$



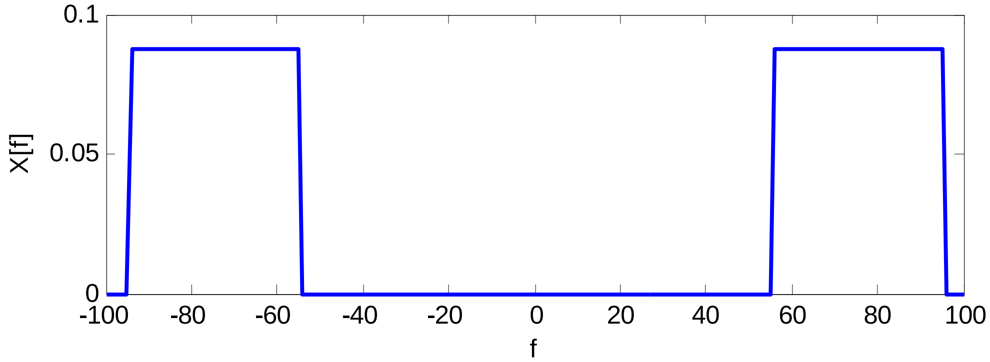
From the frequency domain analysis, we will see this signal is not band limited, meaning f_m is infinite, so the Nyquist rate ($f_{Nyq} = 2f_m$) is infinite.

(b) $x(t) = 7\,\text{sinc}(40t)\cos(150\pi t)$

$$X[f] = \frac{7}{40}\text{rect}\left(\frac{f}{40}\right) * \frac{1}{2}\left[\delta(f-75)+\delta(f+75)\right]$$

$$= \frac{7}{80}\left[\text{rect}\left(\frac{f-75}{40}\right)+\text{rect}\left(\frac{f+75}{40}\right)\right]$$

Shown as:

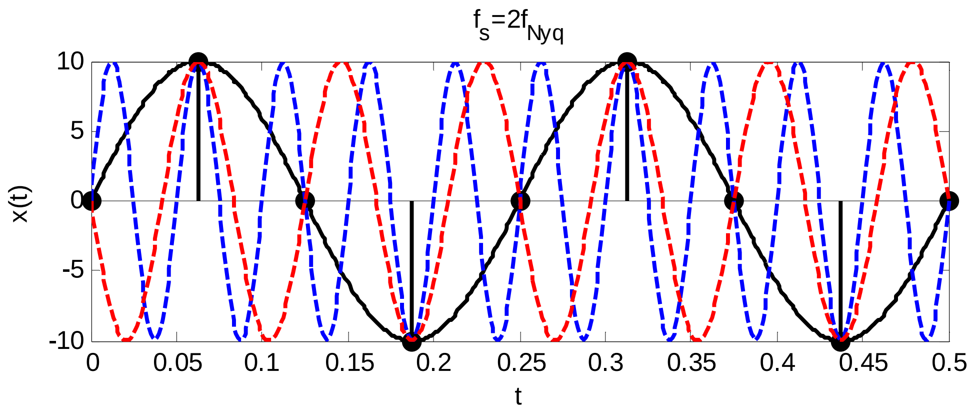


Frequency analysis shows $f_m = 95\text{Hz}$, so the Nyquist rate $f_{Nyq} = 2f_m = 190\text{Hz}$.

34. Signal: $x(t) = 10\sin(8\pi t)$

$$T_0 = 1/4s\,;\quad f_0 = 4\text{Hz}\,;\quad f_{Nyq} = 8\text{Hz}\,;$$

As shown in figure (solid, black), and $x[n]$ formed by sampling $x(t)$ at $f_s = 2f_{Nyq} = 16\text{Hz}$ (black).



In order to yield exactly the same samples when sampled at the same times, there are two cases:

① $x(t) = 10\sin(2\pi f_{k1} t)$, where $f_{k1} = kf_s + f_0$, k is an integer.

As shown in the figure (dashed, blue) for $k = 1$

② $x(t) = -10\sin(2\pi f_{k2} t)$, where $f_{k2} = kf_s - f_0$, k is an integer.

As shown in the figure (dashed, red) for $k = 1$

CODE:

```
clear all,
t = 0:0.001:0.5;
y0 = 10*sin(8*pi*t);
figure(1),subplot(2,1,1),plot(t,y0,'-k')
xlabel('t'),ylabel('x(t)'),hold on
t1 = 0:1/16:0.5;
y1 = 10*sin(8*pi*t1);    %Sample at twice Nyquist rate
figure(1),subplot(2,1,1),stem(t1,y1,'fill','k');
title('f_s=2f_{Nyq}'),
y1_1 = 10*sin(40*pi*t);    %T1=1/20, yields the same sample as original one
figure(1),subplot(2,1,1),plot(t,y1_1,'--b');
y1_2 = -10*sin(24*pi*t);    %T1=1/12, yields the same sample as original one
figure(1),subplot(2,1,1),plot(t,y1_2,'--r'); hold off
```

39. (a) $x(t) = 8 + 3\cos(8\pi t) + 9\sin(4\pi t)$

The period of this signal should be the least common multiple of $\frac{1}{4}$ (the period of $\cos(8\pi t)$) and $\frac{1}{2}$ (the period of $\sin(4\pi t)$), which yields $T_0 = \frac{1}{2}s$. By frequency analysis, we can find $f_m = 4\text{Hz}$, so $f_{Nyq} = 2f_m = 8\text{Hz}$. In order to exactly describe the signal, sampling frequency f_s should larger than Nyquist rate f_{Nyq}. Within a period of $\frac{1}{2}s$, sample values should larger than $T_0 f_{Nyq} = \frac{1}{2} \cdot 8 = 4$ and must be a integer, which yields to 5. So we need 5 samples within one period of T_0 and $f_s = 5/T_0 = 10\text{Hz}$.

(b) $x(t) = 8 + 3\cos(7\pi t) + 9\sin(4\pi t)$

Similar to the part (a), The period T_0 is the least common multiple of $\frac{2}{7}$ (the period of $\cos(7\pi t)$) and $\frac{1}{2}$ (the period of $\sin(4\pi t)$), which yields $T_0 = 2s$. By frequency analysis, we can find $f_m = 3.5\text{Hz}$, so $f_{Nyq} = 2f_m = 7\text{Hz}$. In order to exactly describe the signal, sampling frequency f_s should larger than Nyquist rate f_{Nyq}. Within a period of $2s$, sample values should larger than $T_0 f_{Nyq} = 2 \cdot 7 = 14$ and must be a integer, which yields to 15. So we need 15 samples within one period of $2s$ and $f_s = 15/T_0 = 7.5\text{Hz}$.

44. $$x(t) = 15\cos(300\pi t) + 40\sin(200\pi t)$$

The period of the signal is: $T_0 = \text{LCM}\left\{\frac{1}{150}, \frac{1}{100}\right\} = \frac{1}{50} s$

$f_m = 150\text{Hz}$, $f_{Nyq} = 2f_m = 300\text{Hz}$

(a) $f_s = f_{Nyq} = 300\text{Hz}$

Number of samples within one period is: $N_0 = T_0 f_s = \frac{1}{50} \cdot 300 = 6$

So, sampling $x(t)$ at $t = \frac{n}{N_0} T_0 = \frac{n}{300}$, $n = 0,1,2,3,4,5$, as shown in the figure.

$x[n]$: [15, 19.64, -19.64, -15, 49.64, -49.64]

DFT: $$X_{DFT}[k] = \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi nk}{N_0}}$$

$\Rightarrow X_{DFT}[k]$: [0, 0, -j120, 90, j120, 0]

This can also be calculated in Matlab by commend "fft(x_n)"

CTFS: $$X_{CTFS}[k] = \frac{X_{DFT}[k]}{N_0}$$

Note that calculate $X_{CTFS}[k]$ for k from $-N_0/2$ to $N_0/2$, while $X_{DFT}[k]$ is for k from 0 to $N_0 - 1$, so periodicity of $X_{DFT}[k]$ is used here. The starting point at $-N_0/2$ and ending point at $N_0/2$ are actually repeated, which means the last point in one period of N_0 will overlap the first point in the next period. We need to ensure value after such overlap still fix at $\frac{X_{DFT}[N_0/2]}{N_0}$, so

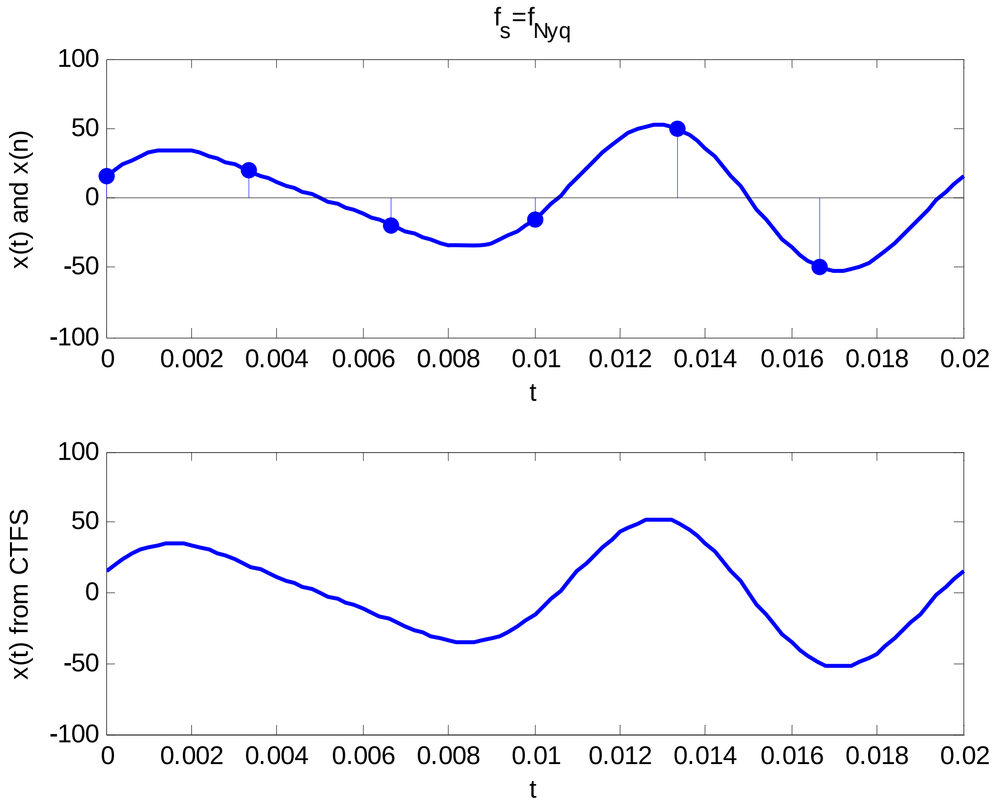
$$X_{CTFS}\left[-\frac{N_0}{2}\right] = X_{CTFS}\left[\frac{N_0}{2}\right] = \frac{X_{DFT}[N_0/2]}{2N_0}$$

$\Rightarrow X_{CTFS}[k]$: [7.5, j20, 0, 0, 0, -j20, 7.5]

The CTFS representation of the signal is:

$$x(t) = \sum_{k=-N_0/2}^{N_0/2} X_{CTFS}[k]e^{j2\pi(kf_0)t} = \sum_{k=-3}^{3} X_{CTFS}[k]e^{j100\pi kt}$$

As shown in the figure, the CTFS representation of the signal looks same as the original signal.



CODE:

```
clear all,
N0 = 6;                              % 6 samples within one period
n = 0:1/50/N0:1/50-1/50/N0;
x_n = 15*cos(300*pi*n)+40*sin(200*pi*n),
x_DFT = fft(x_n),          % get DFT of the samples
x_CTFS = 1/N0*[x_DFT(4)/2,x_DFT(5:6),x_DFT(1),x_DFT(2:3),x_DFT(4)/2], % get CTFS of the samples
t = 0:1/50/100:1/50;
x_t1 = 15*cos(300*pi*t)+40*sin(200*pi*t);
x_t2 = 0;
for nn = -N0/2:N0/2;
x_t2 = x_t2+x_CTFS(nn+N0/2+1)*exp(j*(nn)*2*pi*50*t);
end
figure(1),subplot(2,1,1),plot(t,x_t1)      %original x(t)
title('f_s=f_{Nyq}'),xlabel('t'),ylabel('x(t) and x(n)'), hold on
subplot(2,1,1),stem(n,x_n,'fill'),hold off        %samples x(n)
```

```
figure(1),subplot(2,1,2),plot(t,x_t2)      %CTFS representation of the signal
xlabel('t'),ylabel('x(t) from CTFS'),
```

(b) $f_s = 2f_{Nyq} = 600\text{Hz}$

Number of samples within one period is: $N_0 = T_0 f_s = \frac{1}{50} \cdot 600 = 12$

Sampling $x(t)$ at $t = \frac{n}{N_0} T_0 = \frac{n}{600}$, $n = 0, \cdots, N_0 - 1$, as shown in the figure.

$x[n]$: [15, 34.64, 19.64, 0, -19.64, -34.64, -15, 34.64, 49.64, 0, -49.6410, -34.64]

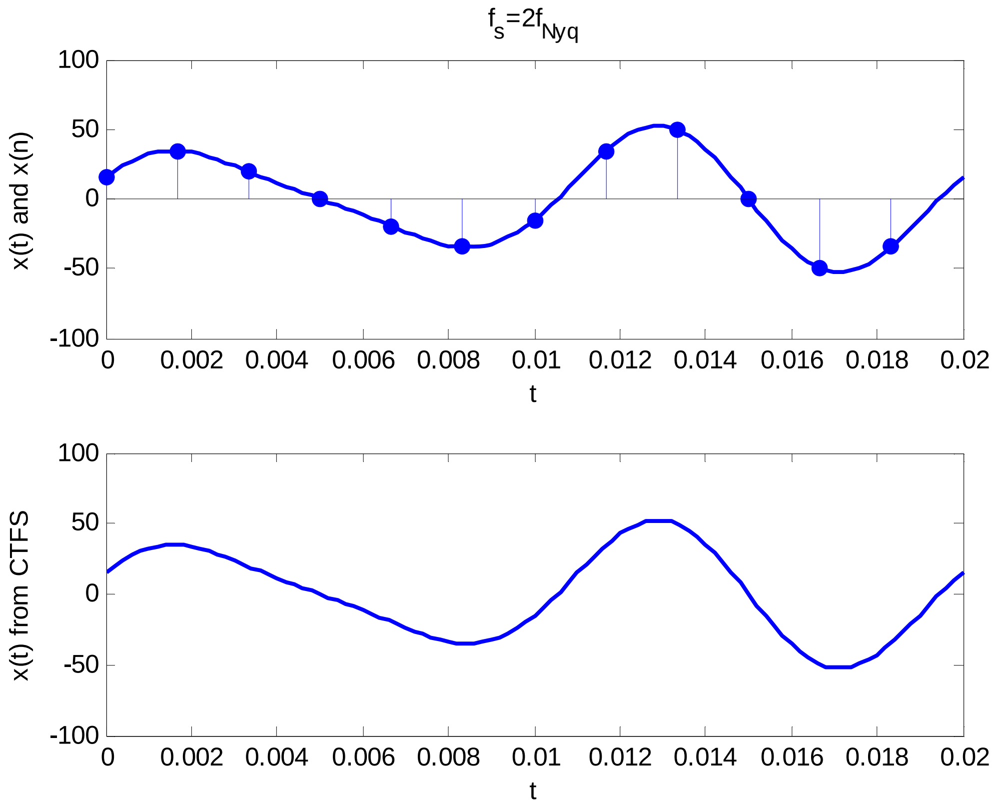
$X_{DFT}[k]$: [0, 0, -j240, 90, 0, 0, 0, 0, 0, 90, j240, 0]

$X_{CTFS}[k]$: [0, 0, 0, 7.5, j20, 0, 0, 0, -j20, 7.5, 0, 0, 0]

The CTFS representation of the signal is:

$$x(t) = \sum_{k=-N_0/2}^{N_0/2} X_{CTFS}[k] e^{j2\pi(kf_0)t} = \sum_{k=-6}^{6} X_{CTFS}[k] e^{j100\pi kt}$$

As shown in the figure, the CTFS representation of the signal looks same as the original signal.



CODE:

```
clear all,
N0 = 12;                    % 12 samples within one period
n = 0:1/50/N0:1/50-1/50/N0;
x_n = 15*cos(300*pi*n)+40*sin(200*pi*n),
x_DFT = fft(x_n),       % get DFT of the samples
x_CTFS = 1/N0*[x_DFT(7)/2,x_DFT(8:12),x_DFT(1),x_DFT(2:6),x_DFT(7)/2],
% get CTFS of the samples
t = 0:1/50/100:1/50;
x_t1 = 15*cos(300*pi*t)+40*sin(200*pi*t);
x_t2 = 0;
for nn = -N0/2:N0/2;
x_t2 = x_t2+x_CTFS(nn+N0/2+1)*exp(j*(nn)*2*pi*50*t);
end
figure(2),subplot(2,1,1),plot(t,x_t1)    %original x(t)
title('f_s=2f_{Nyq}'),xlabel('t'),ylabel('x(t) and x(n)'), hold on
subplot(2,1,1),stem(n,x_n,'fill'),hold off      %samples x(n)
figure(2),subplot(2,1,2),plot(t,x_t2)    %CTFS representation of the signal
xlabel('t'),ylabel('x(t) from CTFS'),
```