Continuous-time Signals

(AKA *analog* signals)

I. Analog* Signals review

Goals:

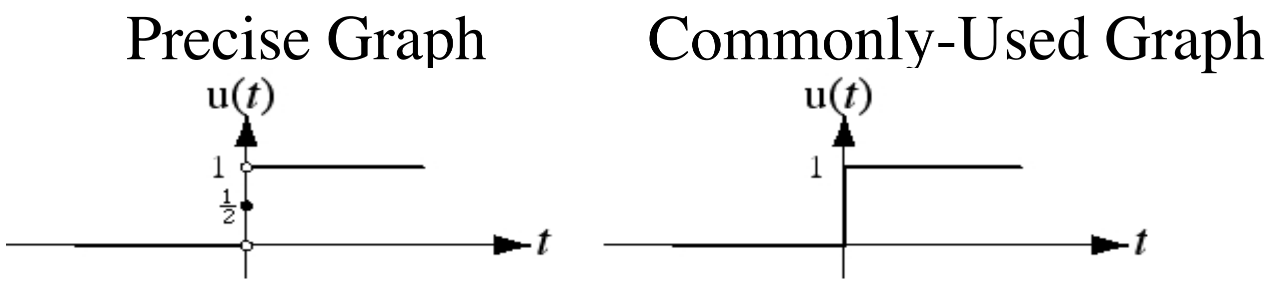
-Common test signals used in system analysis

- Signal operations and properties:
  scaling, shifting, periodicity, energy and power

*Analog = the argument t (time) of the signal $g(t)$ is continuous (CT). Also known as CT signals.

The Unit Step function

$$u(t) = \begin{cases} 1 & , \ t > 0 \\ 1/2 & , \ t = 0 \\ 0 & , \ t < 0 \end{cases}$$



Note: The signal is discontinuous at zero but is an analog signal

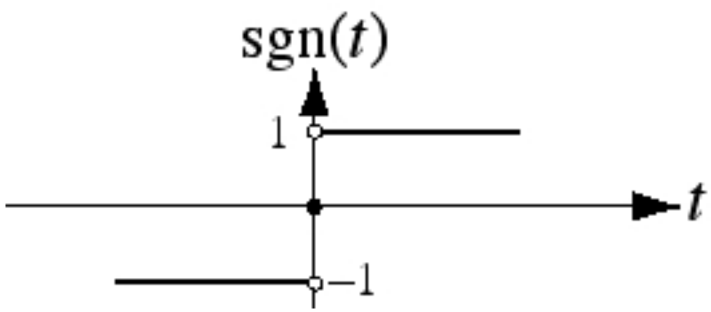
Note: The product signal g(t)u(t) for any g(t) can be thought of as the signal g(t) "turned on" at time t = 0.

Used to check how a system responds to a "sudden" input

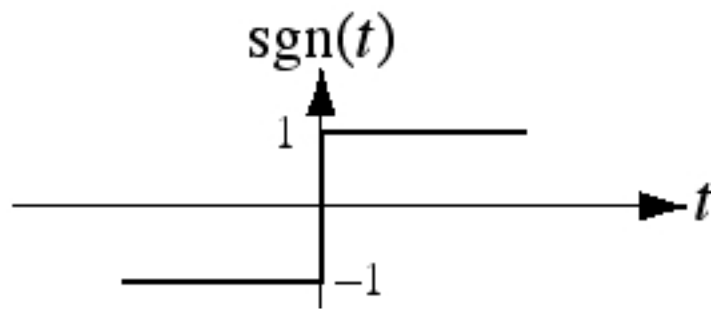
The Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & , \; t > 0 \\ 0 & , \; t = 0 \\ -1 & , \; t < 0 \end{cases} = 2\,\mathrm{u}(t) - 1$$

Precise Graph

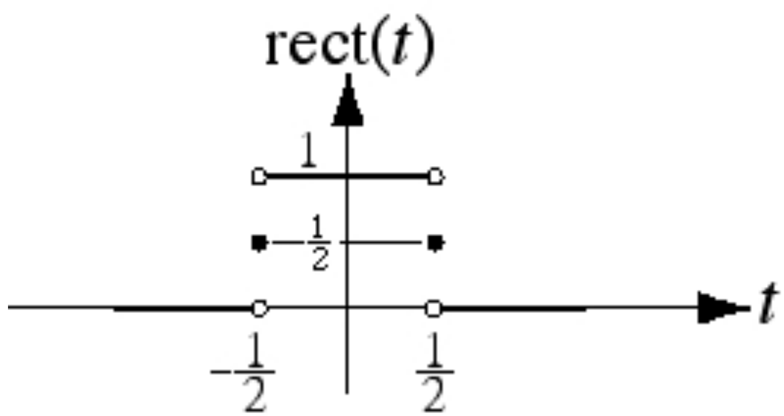


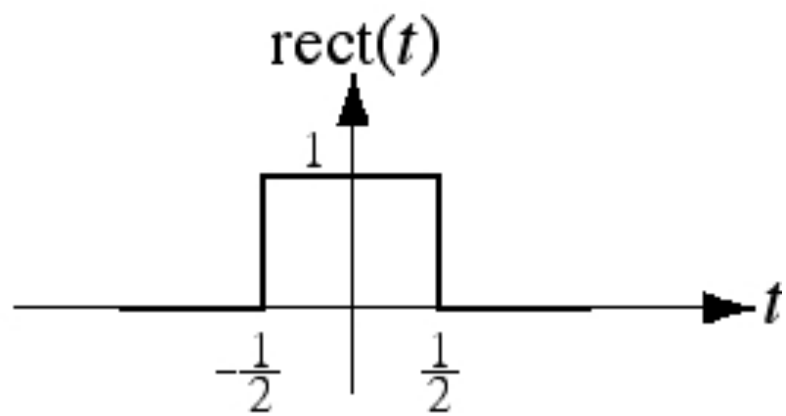
Commonly-Used Graph



The Unit Rectangle function

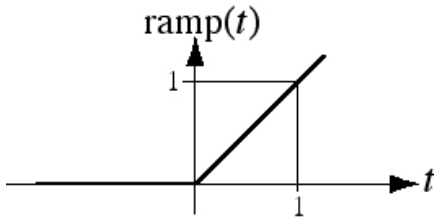
$$\mathrm{rect}(t) = \begin{cases} 1 & , \; |t| < 1/2 \\ 1/2 & , \; |t| = 1/2 \\ 0 & , \; |t| > 1/2 \end{cases} = \mathrm{u}(t + 1/2) - \mathrm{u}(t - 1/2)$$



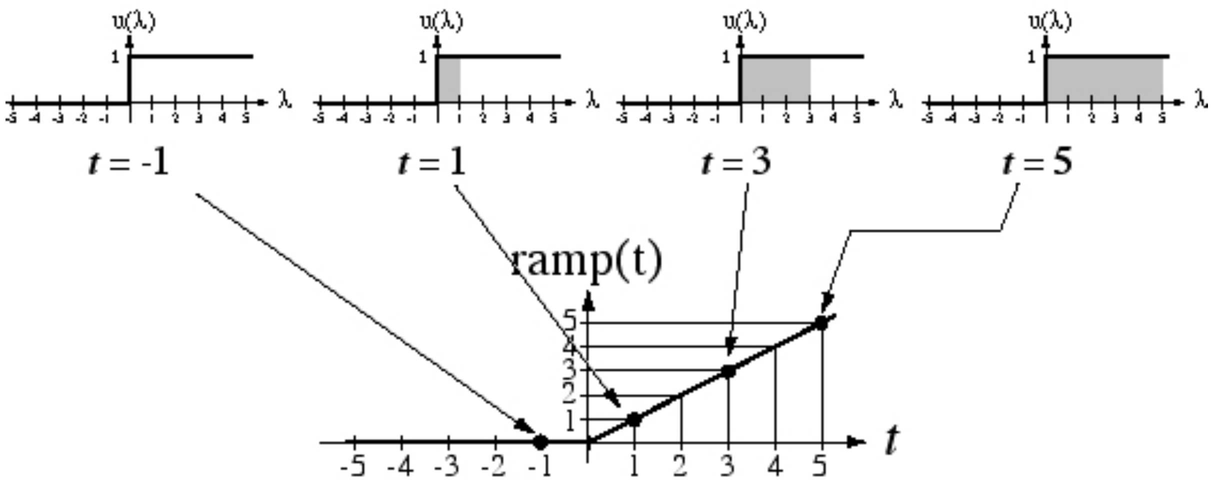


The product $g(t)rect(t)$ can be understood as the signal turned on at $t = -\frac{1}{2}$ and turned off at $t = \frac{1}{2}$

The Unit Ramp function



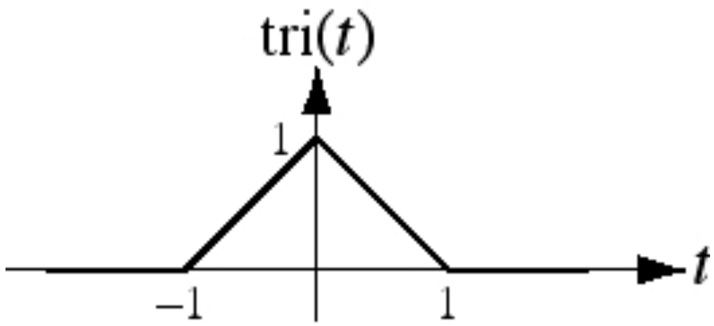
$$\mathrm{ramp}(t) = \begin{cases} t & , \; t > 0 \\ 0 & , \; t \le 0 \end{cases} = \int_{-\infty}^{t} \mathrm{u}(\lambda)\, d\lambda = t\,\mathrm{u}(t)$$



The Unit Ramp function is unbounded with time

Unit Triangle function

$$\text{tri}(t) = \begin{cases} 1 - |t| & , \ |t| < 1 \\ 0 & , \ |t| \geq 1 \end{cases}$$

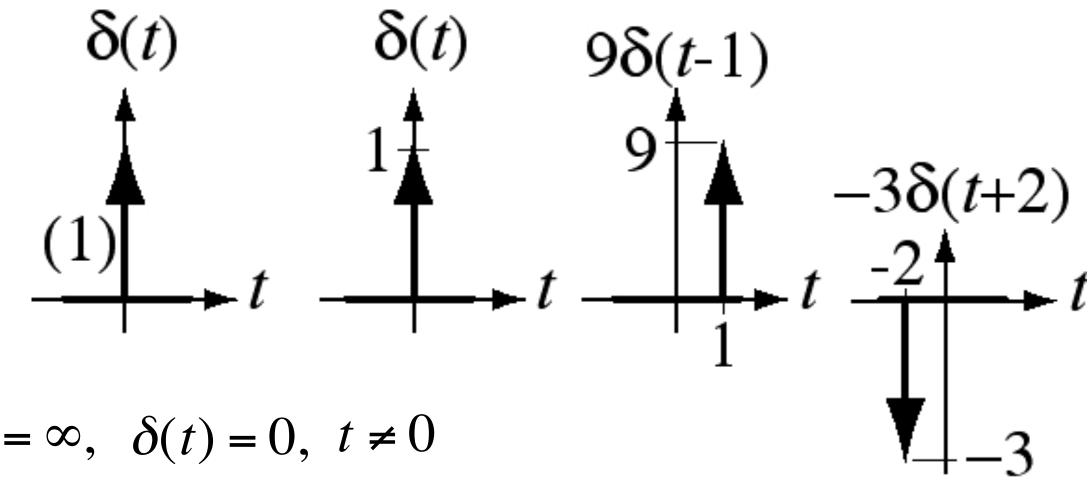


The triangle signal is related to the rectangle through an operation called **convolution**

(to be introduced later…)

The Impulse function

The impulse function is not a function in the ordinary sense because its value at zero is not a real value



$$\delta(0) = \infty, \quad \delta(t) = 0, \quad t \neq 0$$

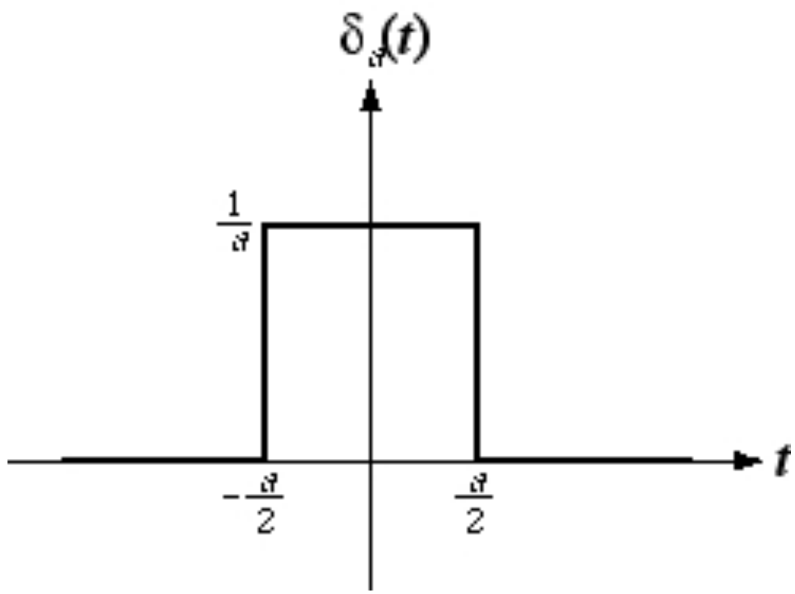
It is represented by a vertical arrow

The impulse function is unbounded and discontinuous

Creating an Impulse

An impulse can be defined as the limit of the rectangle function *with unit area* as a goes to zero

$$\delta_a(t) = \begin{cases} 1/a & , \ |t| < a/2 \\ 0 & , \ |t| > a/2 \end{cases}$$

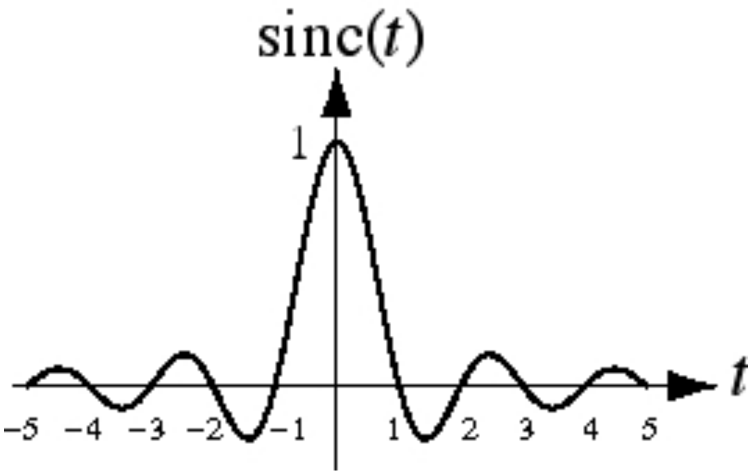


Other approximations are possible…
for example, we can use a triangle function

The Unit Sinc function

The unit sinc function is related to the unit rectangle function through the **Fourier Transform**

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

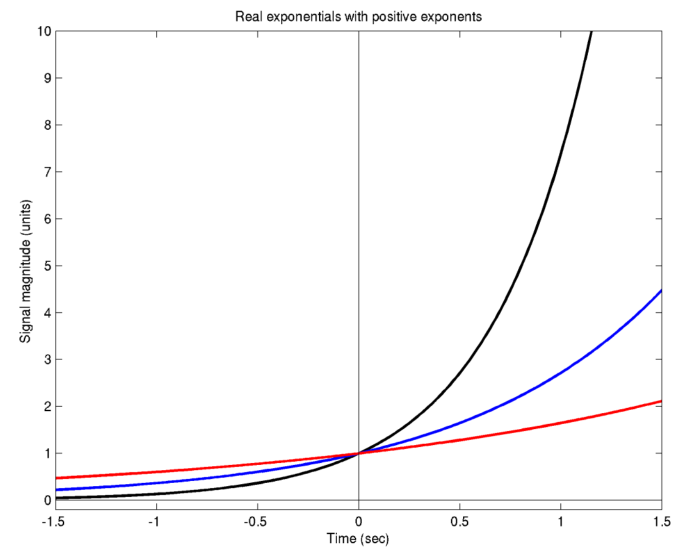


It is used for noise removal in signals

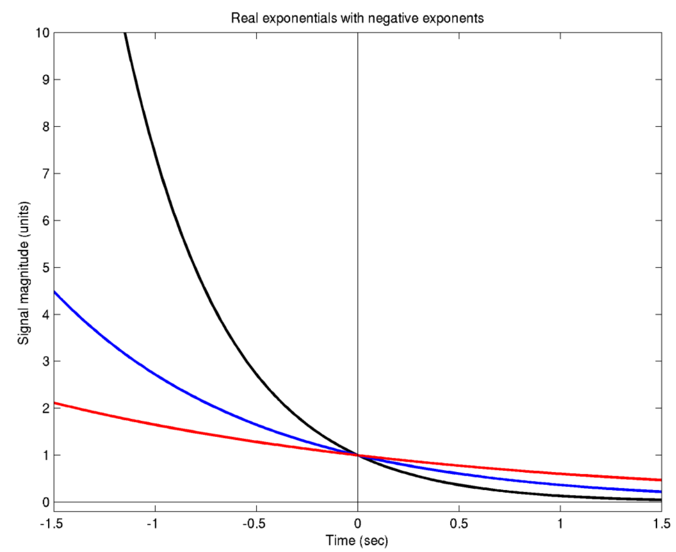
$$\lim_{t\to 0} \operatorname{sinc}(t) = \lim_{t\to 0} \frac{\frac{d}{dt}\left(\sin(\pi t)\right)}{\frac{d}{dt}(\pi t)} = \lim_{t\to 0} \frac{\pi \cos(\pi t)}{\pi} = 1$$

Real exponentials

red $e^{0.5t}$
blue e^{t}
black e^{2t}



red $e^{-0.5t}$
blue e^{-t}
black e^{-2t}



Sinusoids (and Cosinusoids)

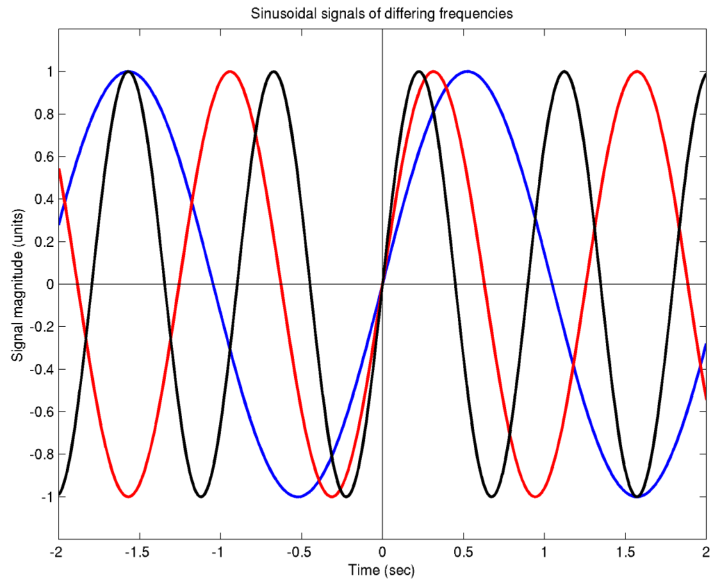
red $\sin(5t)$
blue $\sin(3t)$
black $\sin(7t)$

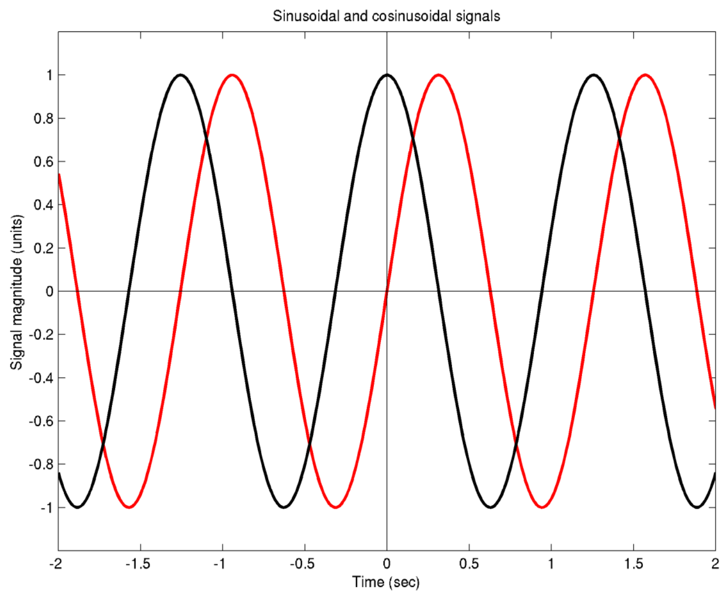
red $\sin(5t)$
black $\cos(5t)$

for

$$\cos(\omega t + \theta) = \cos(2\pi f t + \theta)$$

radian frequency: ω (rad/s)
cyclic frequency: f (cycles/s)
wavelength or period: $1/f$
amplitude: 1
phase: θ



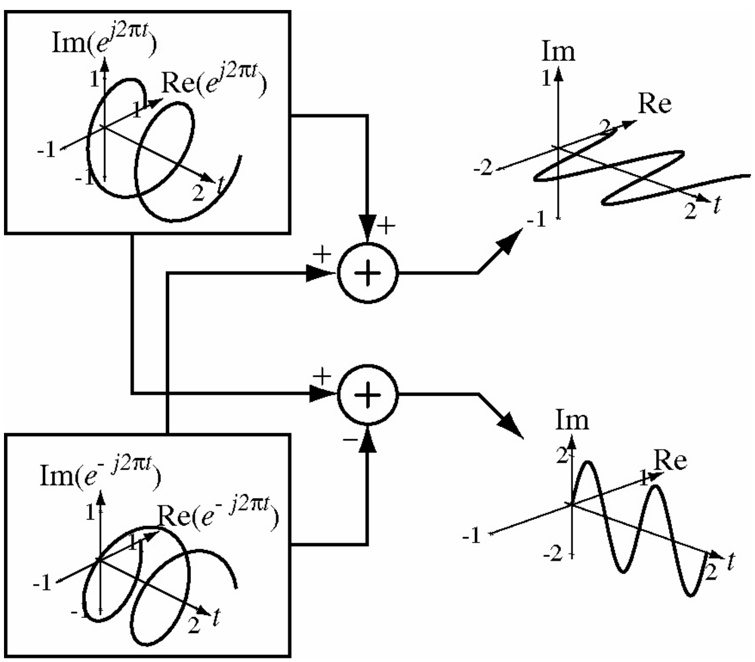


Real and complex sinusoids

Recall that a complex number is defined as $z = a + jb$
(here $\mathrm{Re}(z) = a$ $\mathrm{Im}(z) = b$ and $j = \sqrt{-1}$)

A complex sinusoid is expressed as $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

A graphical example for $\omega = 2\pi$:



Relation between complex and real sinusoids:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Analog signals review

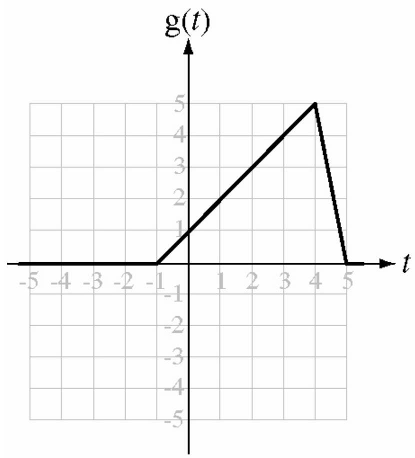
Goals:

- Common test signals used in system analysis

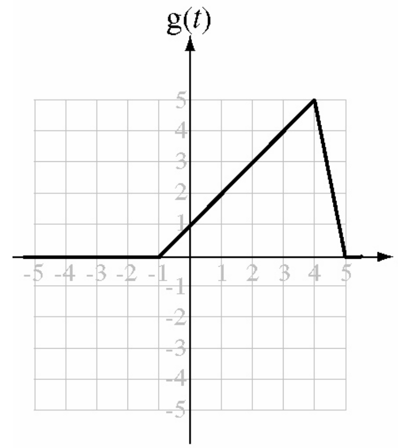
- Signal operations and properties:
  shifting, scaling, periodicity, energy and power

Signal operations: shifting and scaling

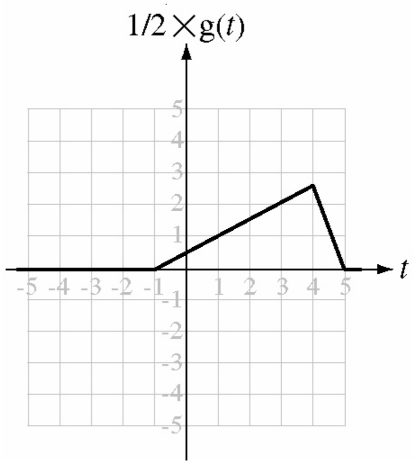
Amplitude scaling $\quad g(t) \rightarrow A\,g(t)$



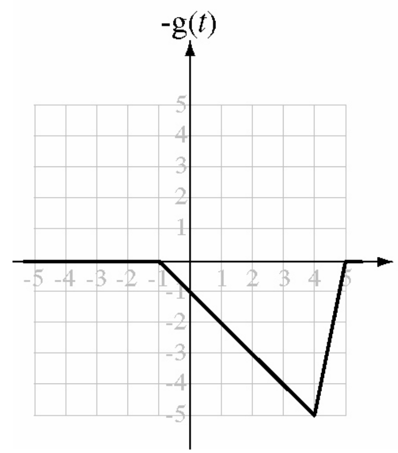
t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0



t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0



t	$1/2 \times g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1/2
1	1
2	3/2
3	2
4	5/2
5	0



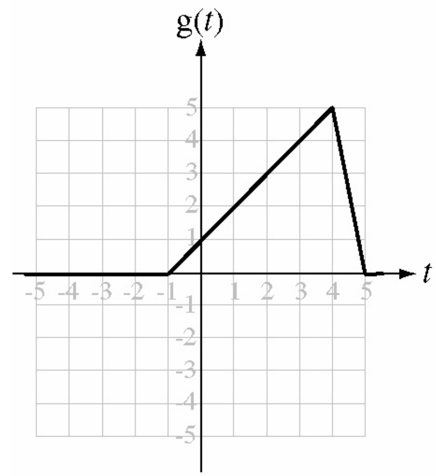
t	$-g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	−1
1	−2
2	−3
3	−4
4	−5
5	0

Expansion/contraction of signal along Y axis
Rotation about X axis for negative A

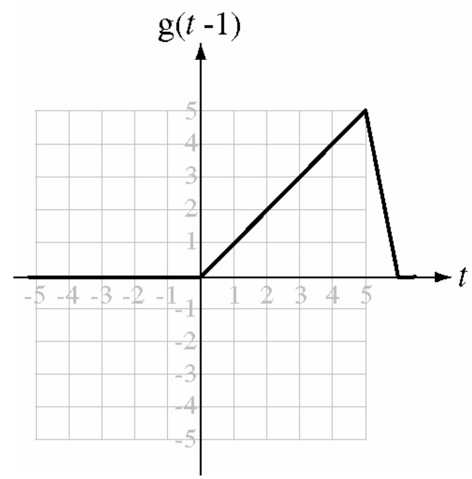
Signal operations: shifting and scaling

Time shifting

Time shifting $t \rightarrow t - t_0$



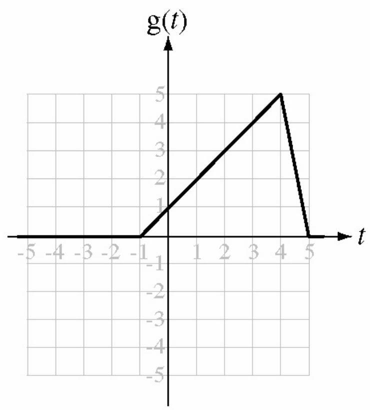
t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0



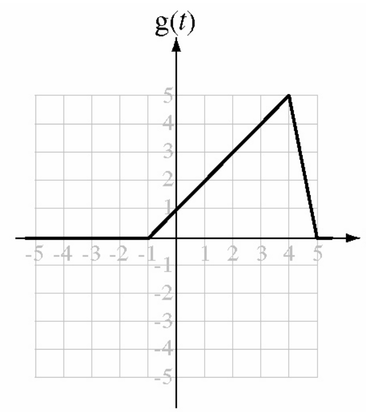
t	$t-1$	$g(t-1)$
−5	−6	0
−4	−5	0
−3	−4	0
−2	−3	0
−1	−2	0
0	−1	0
1	0	1
2	1	2
3	2	3
4	3	4
5	4	5

Signal operations: shifting and scaling

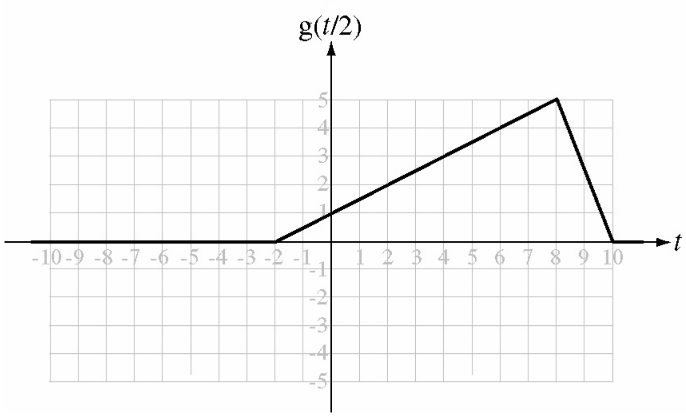
Time scaling $t \rightarrow t/a$



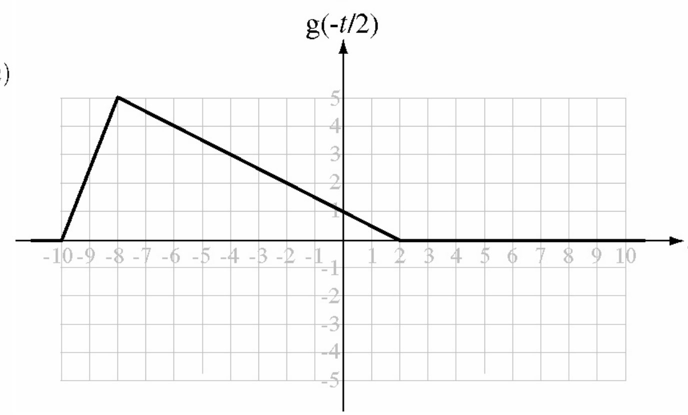
t	$g(t)$
–5	0
–4	0
–3	0
–2	0
–1	0
0	1
1	2
2	3
3	4
4	5
5	0



t	$g(t)$
–5	0
–4	0
–3	0
–2	0
–1	0
0	1
1	2
2	3
3	4
4	5
5	0



t	$t/2$	$g(t/2)$
–10	–5	0
–8	–4	0
–6	–3	0
–4	–2	0
–2	–1	0
0	0	1
2	1	2
4	2	3
6	3	4
8	4	5
10	5	0



t	$-t/2$	$g(-t/2)$
–10	5	0
–8	4	5
–6	3	4
–4	2	3
–2	1	2
0	0	1
2	–1	0
4	–2	0
6	–3	0
8	–4	0
10	–5	0

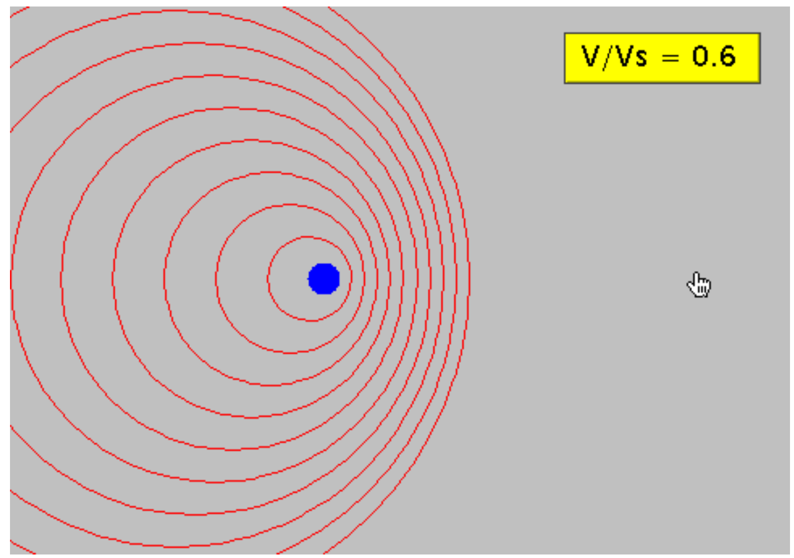
Expansion/contraction of signal along X axis
Rotation about Y axis for negative a

Example: Doppler effect

Sound heard by firefighters: g(t)

Sound we hear when truck comes: A(t) g(at), A increasing, a>1

Sound we hear when truck goes: B(t) g(bt), B decreasing, b<1

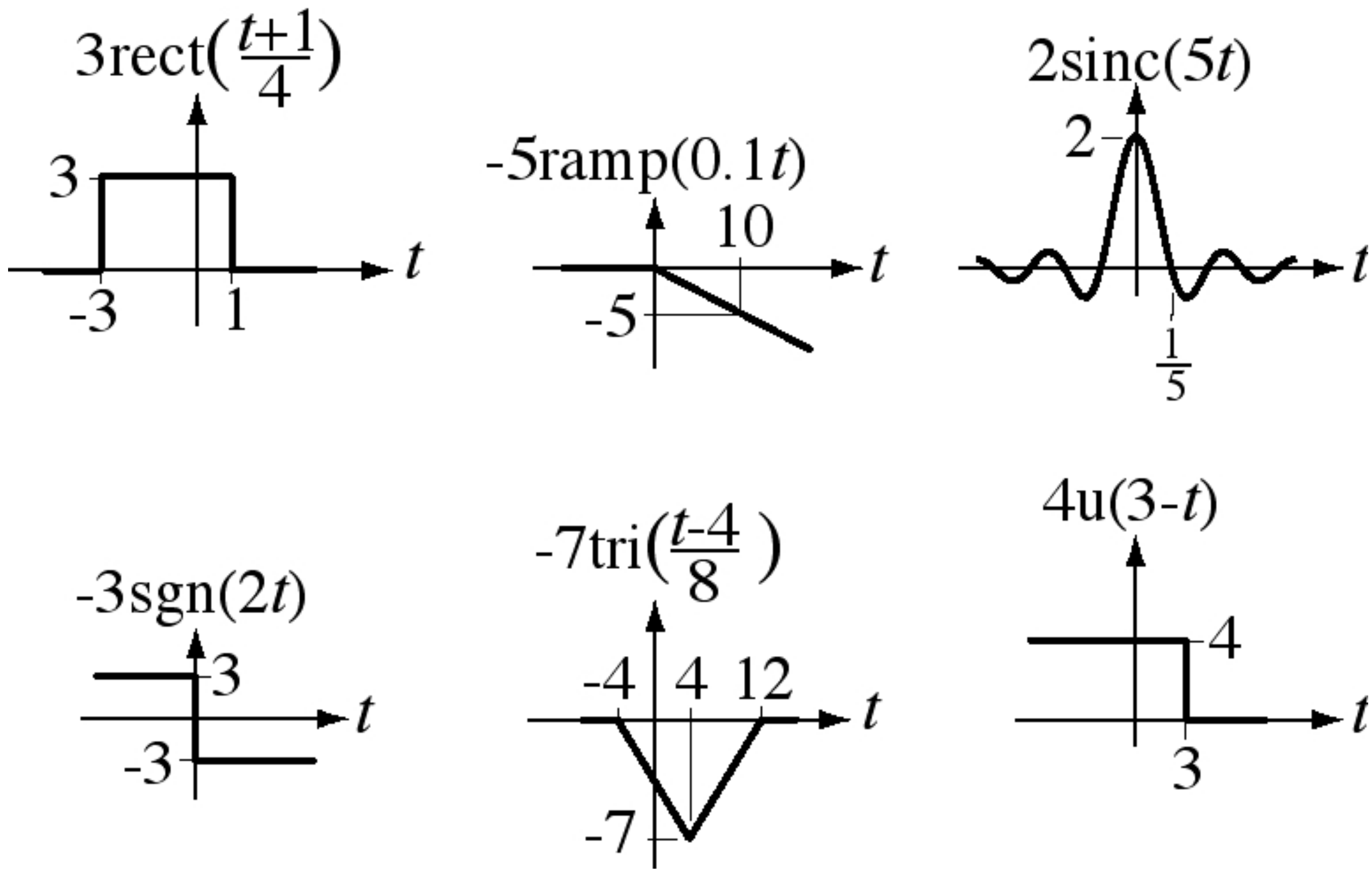


Checkout the website:

http://www.lon-capa.org/~mmp/applist/doppler/d.htm

Signal operations: shifting and scaling

A signal may be expressed by means of basic test signals that have been time-scaled and amplified



Signal operations: shifting and scaling

Caution! The impulse satisfies special properties

Time-scaling property

$$\delta\left(a\left(t-t_0\right)\right)=\frac{1}{|a|}\delta\left(t-t_0\right)$$

Sampling property

$$\int_{-\infty}^{\infty} \mathrm{g}\left(t\right)\delta\left(t-t_0\right)dt=\mathrm{g}\left(t_0\right)$$

The impulse function

is not a function in the ordinary sense

Signal properties: periodic signals

Periodic signals repeat

$$x(t + kT) = x(t)$$

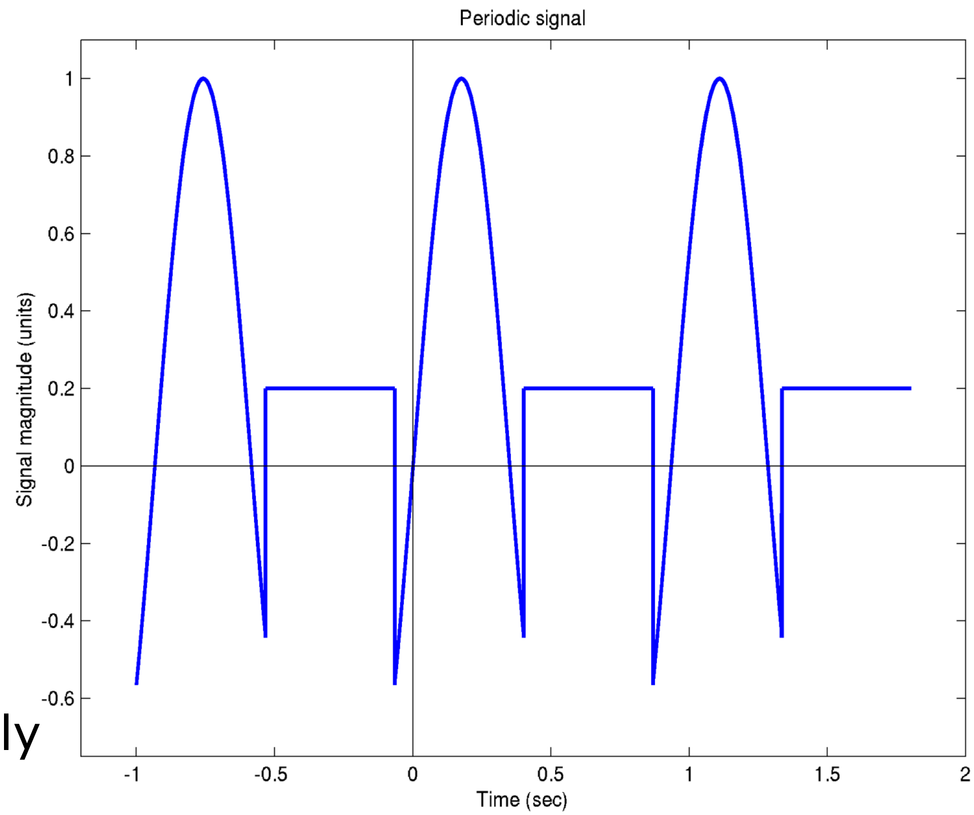
Cycle time is the period T

- Here about 1 second
- 2 seconds is also a period

We only need define the signal over one period and we know everything about it

Sinusoids and constant are clearly periodic signals

Other examples include periodic pulses (rectangular and triangular pulses)



Signal properties: periodic signals

The sum of two periodic signals

$$x(t) = x_1(t) + x_2(t)$$

with periods T_1 and T_2 is:

- periodic when $\frac{T_1}{T_2}$ is rational. If $T_1 = \frac{a_1}{b_1}$, $T_2 = \frac{a_2}{b_2}$ then the period is

$$T = \frac{LCM(a_1, a_2)}{GCD(b_1, b_2)}$$

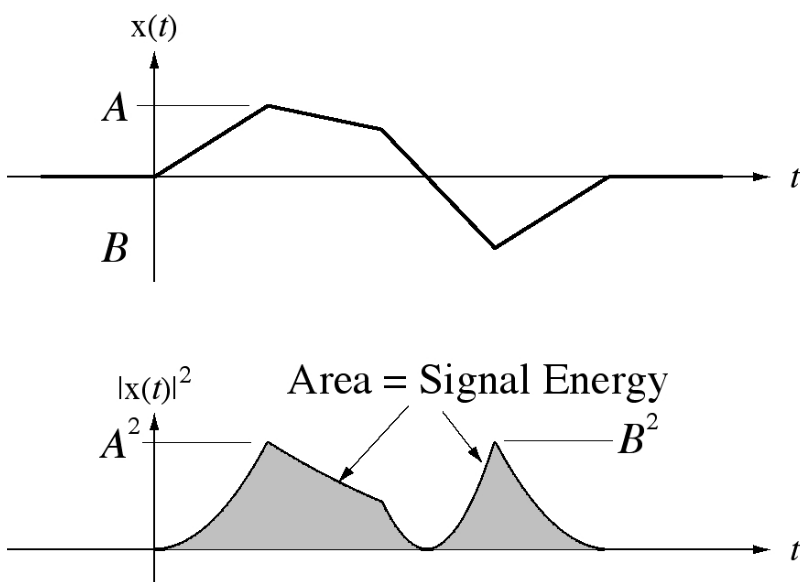
- not periodic when $\frac{T_1}{T_2}$ is irrational

Signal energy and power

Quantifying the "size" of a signal is important in many applications: How much electricity can be used in a defibrillator? How much energy should an audio signal have to be heard?

The energy of the signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



Signal energy and power

Some signals have infinite energy. In that case, we may use the concept of **average signal power**

For a periodic signal, $x(t)$, with period T, the average signal power is

$$P_{\mathrm{x}} = \frac{1}{T}\int_{T} \left|\mathrm{x}(t)\right|^{2} dt$$

If the signal is not periodic, then

$$P_{\mathrm{x}} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\mathrm{x}(t)\right|^{2} dt$$

Analog signals summary

We have seen:

-Signals can be seen as **inputs/outputs** to systems

-Analog signals can be represented as functions of continuous time

-The unit step, impulse, ramp and rectangle functions are **examples of test signals** to systems

-A general signal can be expressed as a combination of some basic test signals by using **scaling/shifting operations**

-Properties of signals include periodicity, even/odd, continuity, differentiability, etc

-Power and energy are concepts that measure signal "size"