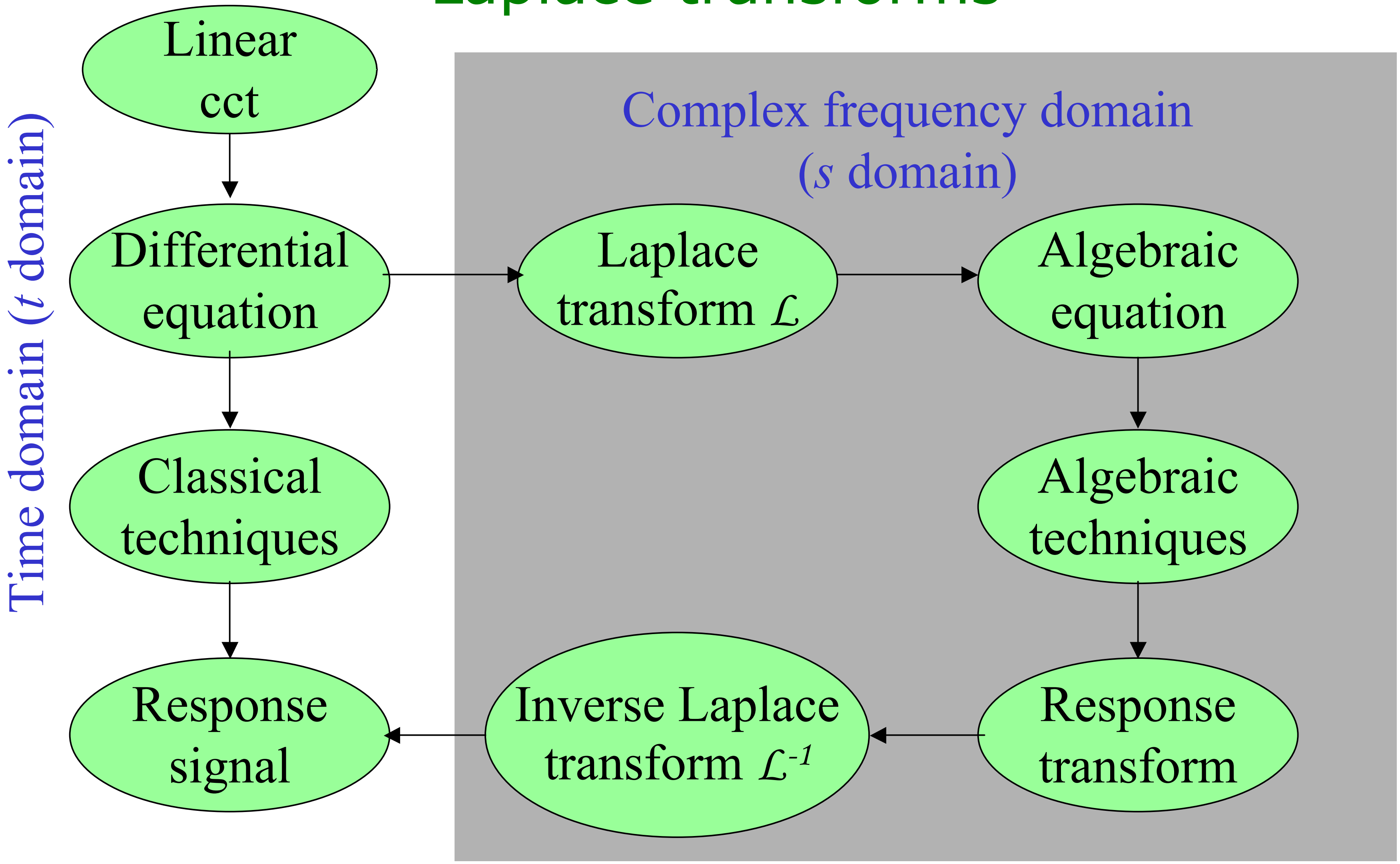
# LTI System Analysis with the Laplace Transform

# Laplace transforms




The diagram commutes

Same answer whichever way you go

# Laplace Transform - definition

## Function *f(t)* of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

*0-* limit is used to capture transients and discontinuities at $t=0$

$s$ is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of $s$ are $sec^{-1}$=Hz

A frequency

# Laplace transform examples

## Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925)

$$u(t) = \begin{cases} 0, \text{ for } t < 0 \\ 1, \text{ for } t \geq 0 \end{cases}$$

$$F(s) = \int_{0-}^{\infty} u(t)e^{-st}dt = \int_{0-}^{\infty} e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_0^{\infty} = -\frac{e^{-(\sigma+j\omega)t}}{\sigma + j\omega}\bigg|_0^{\infty} = \frac{1}{s} \text{ if } \sigma > 0$$

## Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

$$F(s) = \int_{0}^{\infty} e^{-\alpha t}e^{-st}dt = \int_{0}^{\infty} e^{-(s+\alpha)t}dt = -\frac{e^{-(s+\alpha)t}}{s+\alpha}\bigg|_0^{\infty} = \frac{1}{s+\alpha} \text{ if } \sigma > \alpha$$

## Delta (impulse) function $\delta(t)$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1 \text{ for all } s$$

# Laplace Transform Pair Tables

| Signal | Waveform | Transform |
|---|---|---|
| impulse | $\delta(t)$ | $1$ |
| step | $u(t)$ | $\frac{1}{s}$ |
| ramp | $tu(t)$ | $\frac{1}{s^2}$ |
| exponential | $e^{-\alpha t}u(t)$ | $\frac{1}{s+\alpha}$ |
| damped ramp | $te^{-\alpha t}u(t)$ | $\frac{1}{(s+\alpha)^2}$ |
| sine | $\sin(\beta t)u(t)$ | $\frac{\beta}{s^2+\beta^2}$ |
| cosine | $\cos(\beta t)u(t)$ | $\frac{s}{s^2+\beta^2}$ |
| damped sine | $e^{-\alpha t}\sin(\beta t)u(t)$ | $\frac{\beta}{(s+\alpha)^2+\beta^2}$ |
| damped cosine | $e^{-\alpha t}\cos(\beta t)u(t)$ | $\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$ |

# Laplace Transform Properties

## Linearity – absolutely critical property

Follows from the integral definition

$$\mathcal{L}\left\{Af_1(t)+Bf_2(t)\right\}=A\mathcal{L}\left\{f_1(t)\right\}+B\mathcal{L}\left\{f_2(t)\right\}=AF_1(s)+BF_2(s)$$

Example

$$\mathcal{L}\left\{A\cos(\beta t)\right\}=\mathcal{L}\left\{\frac{A}{2}\left[e^{j\beta t}+e^{-j\beta t}\right]\right\}=\frac{A}{2}\mathcal{L}\left\{e^{j\beta t}\right\}+\frac{A}{2}\mathcal{L}\left\{e^{-j\beta t}\right\}$$

$$=\frac{A}{2}\frac{1}{s-j\beta}+\frac{A}{2}\frac{1}{s+j\beta}$$

$$=\frac{As}{s^2+\beta^2}$$

# Laplace Transform Properties

## Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof $\quad \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}dt$

Denote $\quad x = \frac{-e^{-st}}{s}$, and $y = \int_0^t f(\tau)d\tau$

so $\quad \frac{dx}{dt} = e^{-st}$, and $\frac{dy}{dt} = f(t)$

Integrate by parts $\quad \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left[\frac{-e^{-st}}{s}\int_0^t f(\tau)d\tau\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$

# Laplace Transform Properties

## Differentiation Property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

Proof via integration by parts again

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[\frac{df(t)}{dt} e^{-st}\right]_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)e^{-st} dt$$

$$= sF(s) - f(0-)$$

Second derivative

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left[\frac{df(t)}{dt}\right]\right\} = s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df}{dt}(0-)$$

$$= s^2 F(s) - sf(0-) - f'(0-)$$

# Laplace Transform Properties

## General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \cdots - f^{(m)}(0-)$$

## Translation properties

$s$-domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$$

$t$-domain translation

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s) \text{ for } a > 0$$

# Laplace Transform Properties

Initial Value Property

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$$

Final Value Property

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Caveats:

- Laplace transform pairs do not always handle discontinuities properly
  - Often get the average value
- Initial value property no good with impulses
- Final value property no good with cos, sin etc

# Laplace Transform Properties

## Multiplication-Convolution Property

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

Critical property for notion of transfer function

More in a bit…

# Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in $s$

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

$p_i$ are the poles and $z_i$ are the zeros of the function

$K$ is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

# How Laplace transforms are born… in MAE 143A

## From linear ODEs:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = b_0 x^{(m)}(t) + b_1 x^{(m-1)}(t) + \ldots + b_m x(t)$$

Under zero initial conditions, Laplace tranforms produce

$$\left(s^n + a_1 s^{n-1} + \ldots + a_n\right) Y(s) = \left(b_0 s^m + b_1 s^{m-1} + \ldots + b_m\right) X(s)$$

For any given X(s) we can compute Y(s)!

$$Y(s) = H(s)X(s), \quad H(s) := \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

This is the **same as convolution** in the time domain:

$$H(s) = \mathcal{L}\left\{h(t)\right\}, \quad Y(s) = H(s)X(s), \quad y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

# Example

Compute the impulse response of the RC circuit described by the ODE:

$$y'(t) + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Apply Laplace tranform:

$$Y(s) = H(s)X(s), \quad H(s) = \frac{\omega_c}{s + \omega_c}, \quad \omega_c := \frac{1}{RC}$$

The impulse response is then

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\} = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

# Inverting Laplace Transforms

We have a table of inverse LTs

Write F(s) as a partial fraction expansion

$$
\begin{aligned}
F(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \\
&= K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \\
&= \frac{\alpha_1}{(s - p_1)} + \frac{\alpha_2}{(s - p_2)} + \frac{\alpha_{31}}{(s - p_3)} + \frac{\alpha_{32}}{(s - p_3)^2} + \frac{\alpha_{33}}{(s - p_3)^3} + \ldots + \frac{\alpha_q}{(s - p_q)}
\end{aligned}
$$

Now appeal to linearity to invert via the table

- Surprise!
- Nastiness: computing the partial fraction expansion is best done by calculating the residues

# Inverting Laplace Transforms

Compute residues at the poles

$$\lim_{s \to a} (s-a)F(s) \qquad\qquad \frac{1}{(j-1)!}\lim_{s \to a}\frac{d^{j-1}}{ds^{j-1}}\left[(s-a)^m F(s)\right]$$

Bundle complex conjugate pole pairs into second-order terms if you want

$$(s-\alpha-j\beta)(s-\alpha+j\beta)=\left[s^2-2\alpha s+\left(\alpha^2+\beta^2\right)\right]$$

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials

In Matlab, check out [r,p,k]=residue(b,a), where

b = coefficients of numerator; a = coefficients of denominator

r = residues; p = poles; k = result of long division

# Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \dfrac{20(s+3)}{(s+1)(s^2+2s+5)}$

$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1-2j} + \frac{k_2^*}{s+1+2j}$$

$$k_1 = \lim_{s \to -1} (s+1)F(s) = \left.\frac{20(s+3)}{s^2+2s+5}\right|_{s=-1} = 10$$

$$k_2 = \lim_{s \to -1+2j} (s+1-2j)F(s) = \left.\frac{20(s+3)}{(s+1)(s+1+2j)}\right|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$f(t) = \left[10e^{-t} + 5\sqrt{2}e^{(-1+2j)t + j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-2j)t - j\frac{5}{4}\pi}\right]u(t)$$

$$= \left[10e^{-t} + 10\sqrt{2}e^{-t}\cos(2t + \frac{5\pi}{4})\right]u(t)$$

# Laplace Transforms with Multiple Poles

## Compute residues at the poles

$$\frac{1}{(j-1)!}\lim_{s\to a}\frac{d^{j-1}}{ds^{j-1}}\left[(s-a)^m F(s)\right]$$

**Example** $$\frac{2s^2+5s}{(s+1)^3}=\frac{2(s+1)^2+(s+1)-3}{(s+1)^3}=\frac{2}{s+1}+\frac{1}{(s+1)^2}-\frac{3}{(s+1)^3}$$

$$\lim_{s\to -1}\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}=-3 \qquad \lim_{s\to -1}\frac{d}{ds}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right]=1$$

$$\frac{1}{2!}\lim_{s\to -1}\frac{d^2}{ds^2}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right]=2$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+5s}{(s+1)^3}\right]=e^{-t}(2+t-3t^2)u(t)$$

# Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \dfrac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$F(s) = s + 2 + \frac{s+2}{s^2 + 4s + 3}$$

$$= s + 2 + \frac{0.5}{s+1} + \frac{0.5}{s+3}$$

Invert as normal

$$f(t) = \left[\delta'(t) + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t}\right]u(t)$$

# System stability

Easy to determine using the transfer function $H(s)$

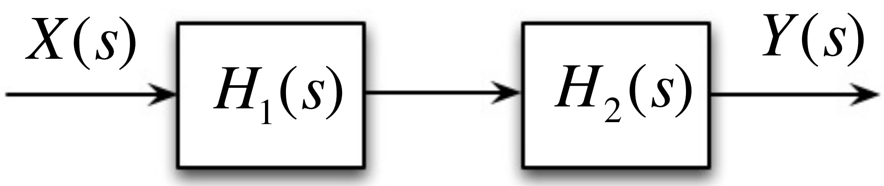
System is BIBO stable if

all the **poles** of the transfer function lie in the **open left half** of the *s* plane

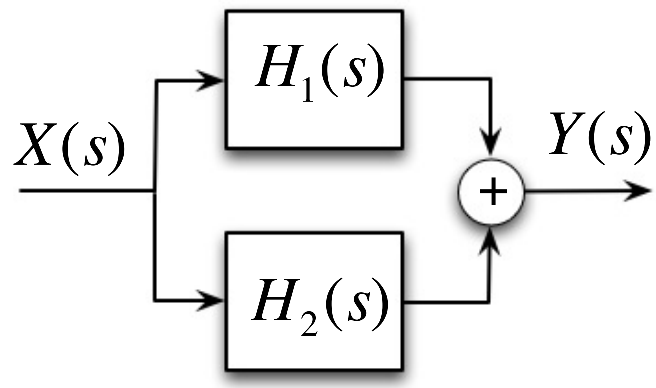
The system is called marginally stable if there are simple poles on the imaginary axis and no poles in the right half-plane.

Marginally stable is a particular case of BIBO unstable.

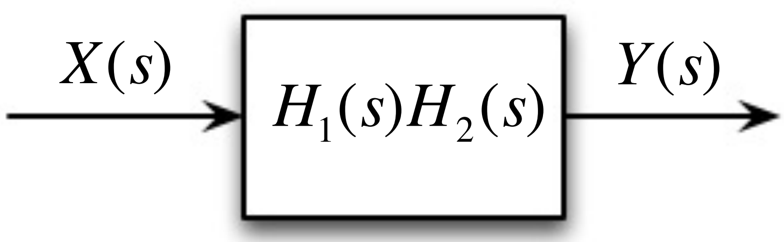
# System interconnections

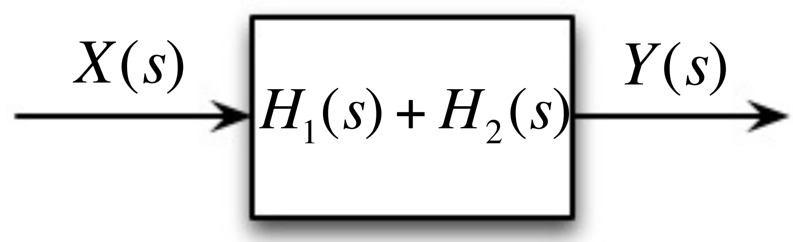



Cascade interconnection

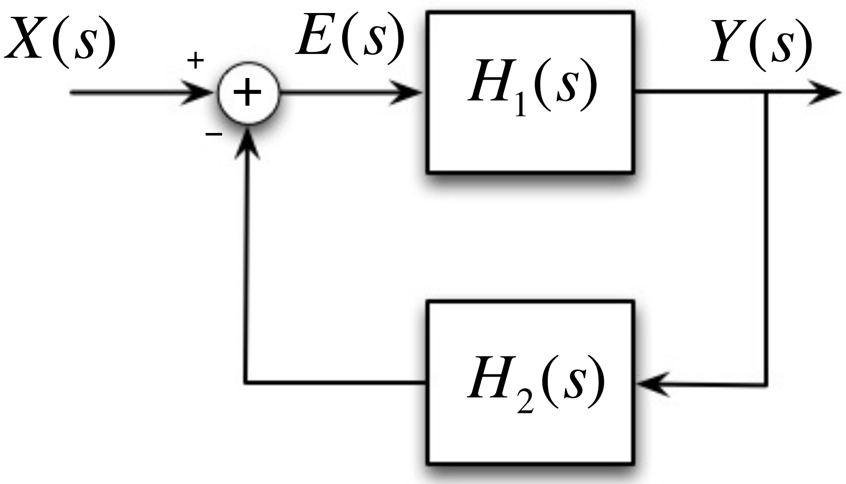


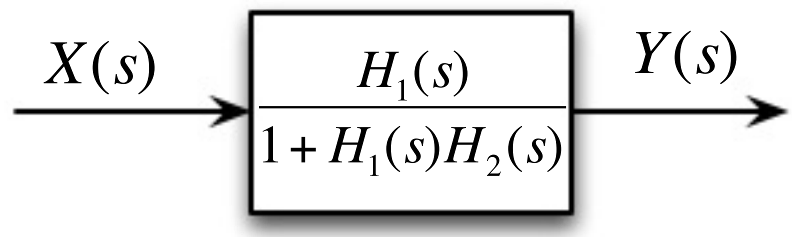

Parallel interconnection



# Feedback interconnection




$H_1(s)$ is the plant or original system we want to control

$H_2(s)$ is the sensor/controller selected to make closed-loop system have certain property (e.g., stability)

Example: $H_1(s) = \frac{1}{s}, \quad H_2(s) = K, \qquad \frac{H_1(s)}{1+H_1(s)H_2(s)} = \frac{1}{s+K}$

Much more about this in MAE143B…

# Block diagrams - revisited

Multiple ways of drawing a system block diagram for strictly proper ($n \geq m$) transfer functions

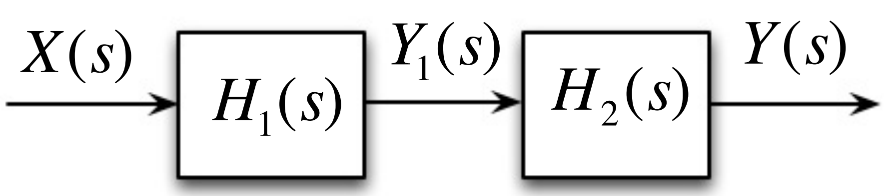
$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

(if $n > m$, then some of the $b_i$ are zero)

- $H(s)$ as a product, $H_1(s) H_2(s)$

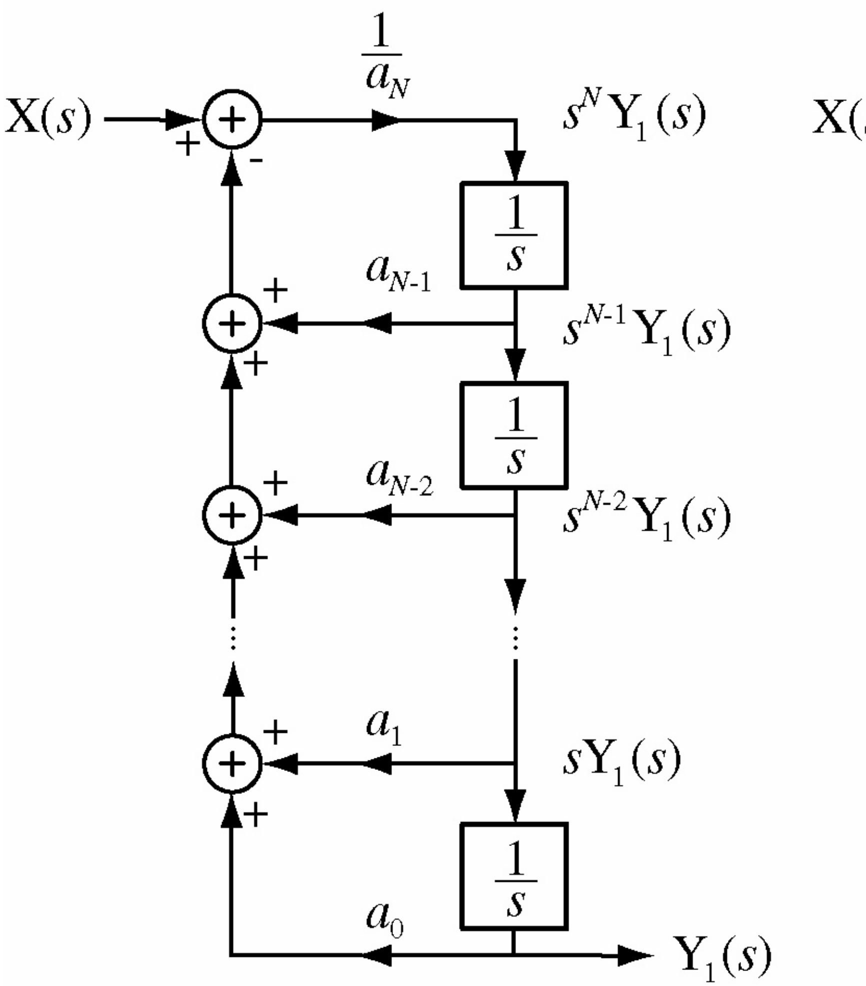
$$H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$H_2(s) = \frac{Y(s)}{Y_1(s)} = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0$$

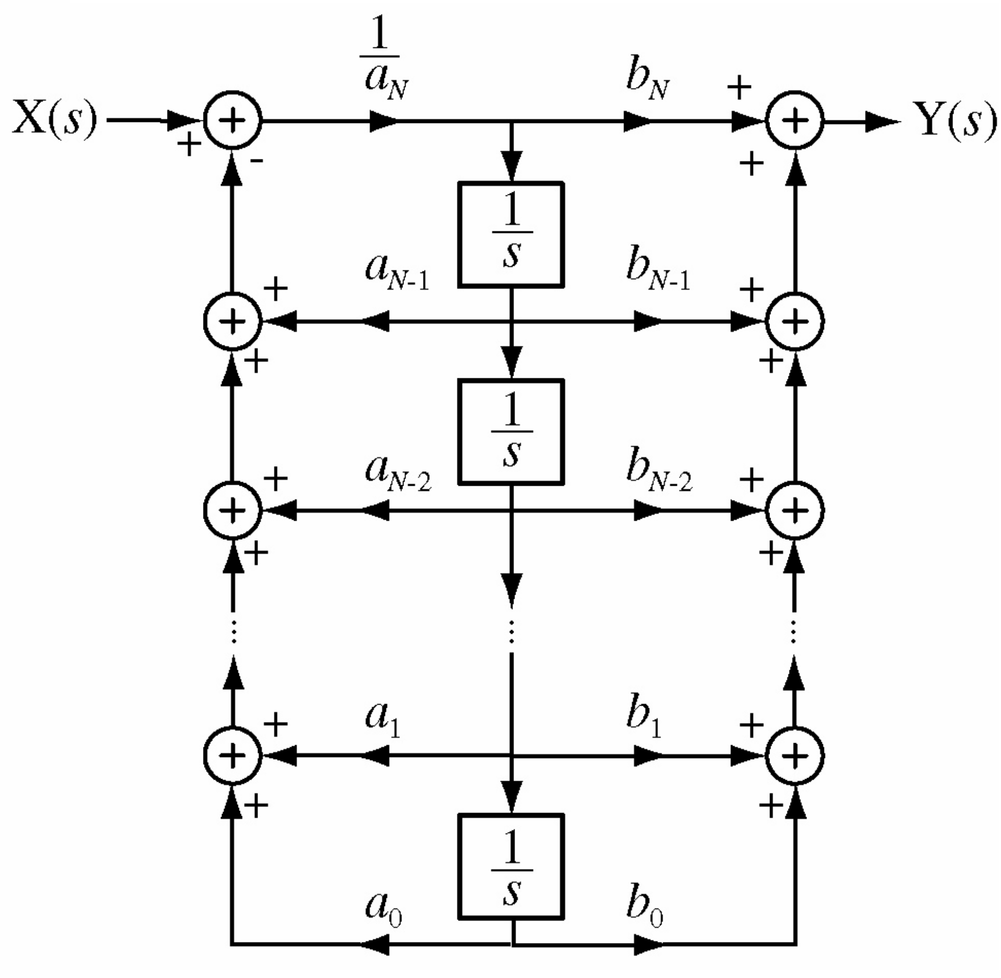


# Block diagrams - revisited

$$s^N \, \mathrm{Y}_1(s) = \frac{1}{a_N}\left\{\mathrm{X}(s) - \left[a_{N-1}s^{N-1}\,\mathrm{Y}_1(s) + \cdots + a_1 s\,\mathrm{Y}_1(s) + a_0\,\mathrm{Y}_1(s)\right]\right\}$$




$H_1(s)$

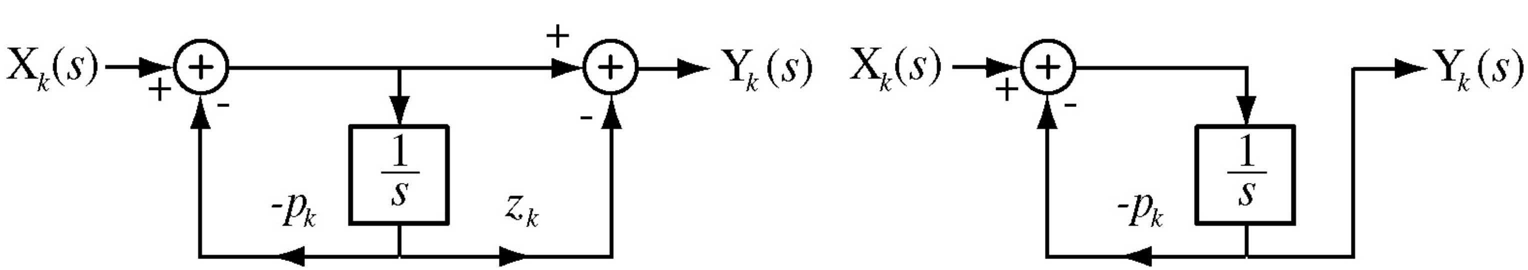



$H(s)$

# Block diagrams - revisited

- in cascade form

$$H(s) = A\frac{(s-z_1)}{(s-p_1)}\cdots\frac{(s-z_m)}{(s-p_m)}\frac{1}{(s-p_{m+1})}\cdots\frac{1}{(s-p_n)}$$




- in parallel form

$$H(s) = \frac{K_1}{(s-p_1)} + \cdots + \frac{K_n}{(s-p_n)}$$