Frequency Response and Continuous-time Fourier Series

Recall course objectives

Main Course Objective:

Fundamentals of systems/signals interaction

(we'd like to understand how systems transform or affect signals)

Specific Course Topics:

- -Basic test signals and their properties
- -Systems and their properties
- -Signals and systems interaction
  - Time Domain: convolution
  - **Frequency Domain: frequency response**
- -Signals & systems applications:
  - audio effects, filtering, AM/FM radio
- -Signal sampling and signal reconstruction

CT Signals and Systems in the FD -part I

Goals

I. Frequency Response of (stable) LTI systems

- -Frequency Response, amplitude and phase definition
- -LTI system response to multi-frequency inputs

II. (stable) LTI system response to periodic signals in the FD

- -The Fourier Series of a periodic signal
- -Periodic signal magnitude and phase spectrum
- -LTI system response to general periodic signals

III. Filtering effects of (stable) LTI systems in the FD

- - Noise removal and signal smoothing

Frequency Response of LTI systems

We have seen how **some specific LTI system responses** (the IR and the step response) can be used to find the response to the system to **arbitrary inputs through the convolution** operation.

However, all **practical (periodic or pulse-like) signals** that can be generated in the lab or in a radio station **can be expressed as superposition of co-sinusoids** with different frequencies, phases, and amplitudes (an oscillatory input is easier to reproduce in the lab than an impulse delta, which has to be approximated)

Because of this, it is of interest to study (stable) LTI system responses to general multi-frequency inputs. This is what defines the frequency response of the system.

We will later see how to use this information to obtain the response of LTI systems to (finite-energy) signals (using Fourier and Laplace transforms)

Response of Systems to Exponentials

Let a stable LTI system be excited by an **exponential input**

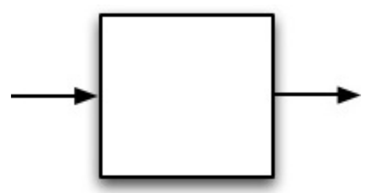
$$x(t) = Ae^{\lambda t}$$

Here, λ and A can be complex numbers!

From what we learned on the response of LTI systems:

The **response** to an exponential **is another exponential**

$$x(t) = Ae^{\lambda t}$$



$$y(t) = Be^{\lambda t}$$

Problem: Determine B

Response of Systems to Exponentials

Consider a linear ODE describing the LTI system as

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Let $x(t) = Ae^{\lambda t}$ and $y(t) = Be^{\lambda t}$

Substituting in the ODE we see that

$$Be^{\lambda t}\sum_{k=0}^{N} a_k \lambda^k = Ae^{\lambda t}\sum_{k=0}^{M} b_k \lambda^k \quad \Rightarrow \quad \boxed{B = \frac{\sum_{k=0}^{M} b_k \lambda^k}{\sum_{k=0}^{N} a_k \lambda^k} A}$$

The proportionality constant is equal to the ratio:

$$\boxed{\frac{B}{A} = \frac{\sum_{k=0}^{M} b_k \lambda^k}{\sum_{k=0}^{N} a_k \lambda^k} = \frac{b_M \lambda^M + \ldots + b_2\lambda^2 + b_1\lambda + b_0}{a_N \lambda^N + \ldots + a_2\lambda^2 + a_1\lambda + a_0}}$$

Response of Systems to Exponentials

Surprise #1:

The ratio

$$\frac{B}{A} = H(\lambda)$$

where

$$H(s) = \frac{b_M s^M + \ldots. + b_2 s^2 + b_1 s + b_0}{a_N s^N + \ldots. + a_2 s^2 + a_1 s + a_0}$$

is the **Transfer Function** which we have obtained before with **Laplace Transforms**!

Response of Systems to Exponentials

Surprise #2:

If $x(t) = A\cos(\omega t) = A\left(e^{j\omega t} + e^{-j\omega t}\right)/2$

using linearity

$$y(t) = A\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right)/2$$

With bit of complex algebra we write

$$H(j\omega) = Ce^{j\theta} \qquad H(-j\omega) = Ce^{-j\theta}$$

and the response

$$y(t) = A\left(Ce^{j\omega t+\theta} + Ce^{-j\omega t-\theta}\right)/2 = AC\cos(\omega t + \theta)$$

where

$$C = \left|H(j\omega)\right| \qquad \theta = \angle H(j\omega)$$

Response of Systems to Exponentials

The ratio $$\frac{B}{A} = H(j\omega) = \frac{b_M (j\omega)^M + ... + b_2 (j\omega)^2 + b_1 (j\omega) + b_0}{a_N (j\omega)^N + ... + a_2 (j\omega)^2 + a_1 (j\omega) + a_0}$$

is called the **Frequency Response**

Observe that $H(j\omega)$ is a complex function of ω

$$H(j\omega) = \mathrm{Re}(H(j\omega)) + j\,\mathrm{Im}(H(j\omega))$$

We can graph the FR by plotting:

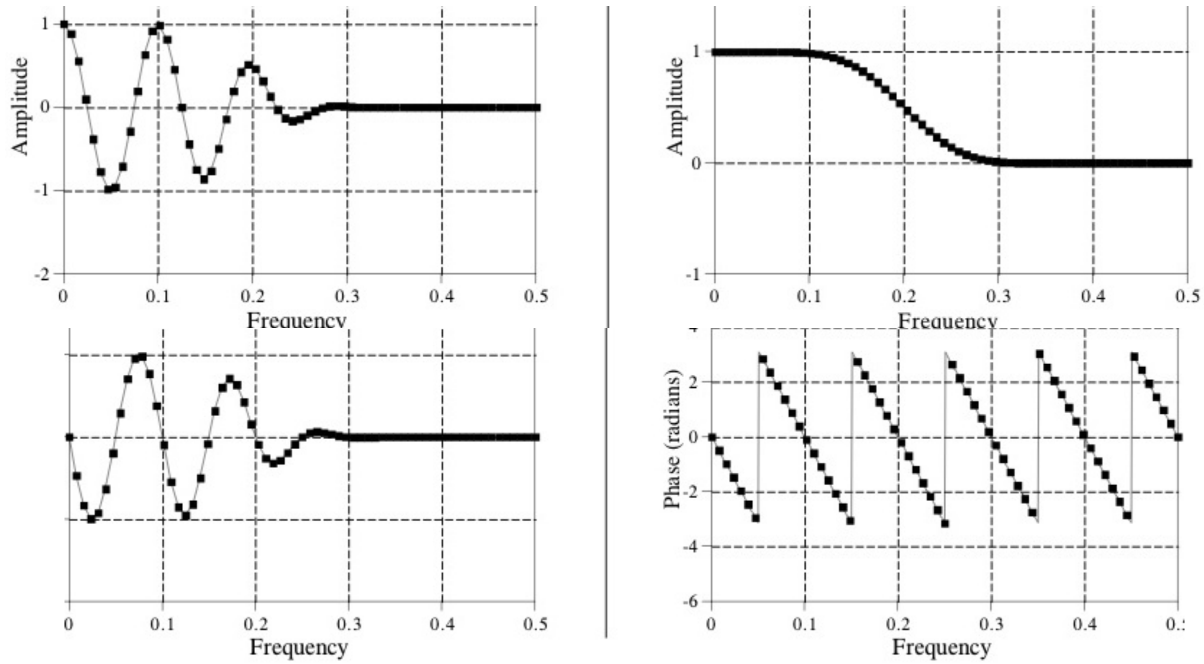
its rectangular coordinates $\mathrm{Re}\, H(j\omega)$, $\mathrm{Im}\, H(j\omega)$ against ω

or its polar coordinates $|H(j\omega)|$, $\angle H(j\omega)$ against ω

Polar coordinates are usually more informative

Frequency Response Plots

Compare rectangular (left) versus polar (right) plots



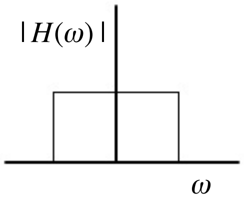
Polar: only low frequency co-sinusoids are passed, the shift in phase is more or less proportional to frequency

Rectangular: contains the same information as polar representation. More difficult to see what it does to co-sinusoids. Mostly used in computer calculations

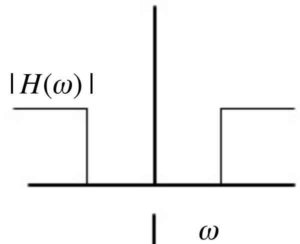
System classification according to FR

Depending on the plot of $|H(\omega)|$ we will classify systems into*:

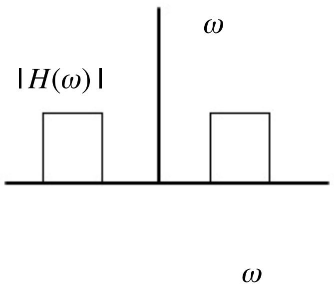
Low-pass filters: $|H(j\omega)| \approx 0$ for $|\omega| > K$



High-pass filters: $|H(j\omega)| \approx 0$ for $|\omega| < K$



Band-pass filters: $|H(j\omega)| \approx 0$ for $|\omega| < K_1$ and $|\omega| > K_2$



In order to plot $|H(j\omega)|$ the logarithmic scale is frequently used. This scale defines decibel units

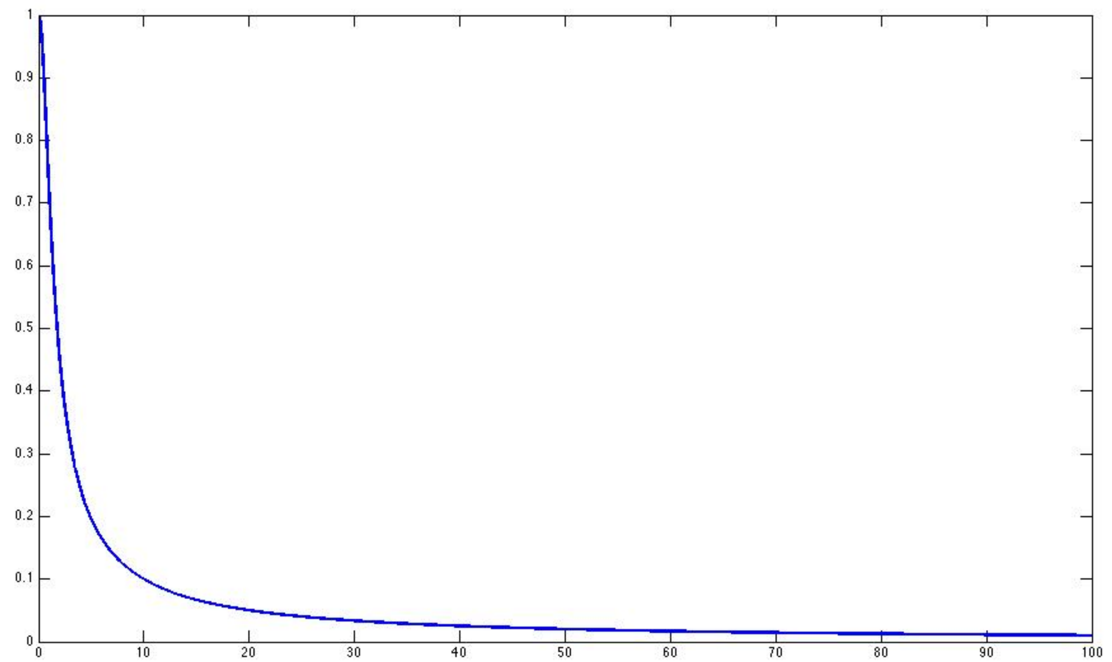
(here we use log_10) $$|H(j\omega)|_{dB} = 20\log|H(j\omega)|$$

(*) these plots correspond to the so-called *ideal filters*, because they keep exactly a set of low, high, and band frequencies

dB representation of Frequency Response

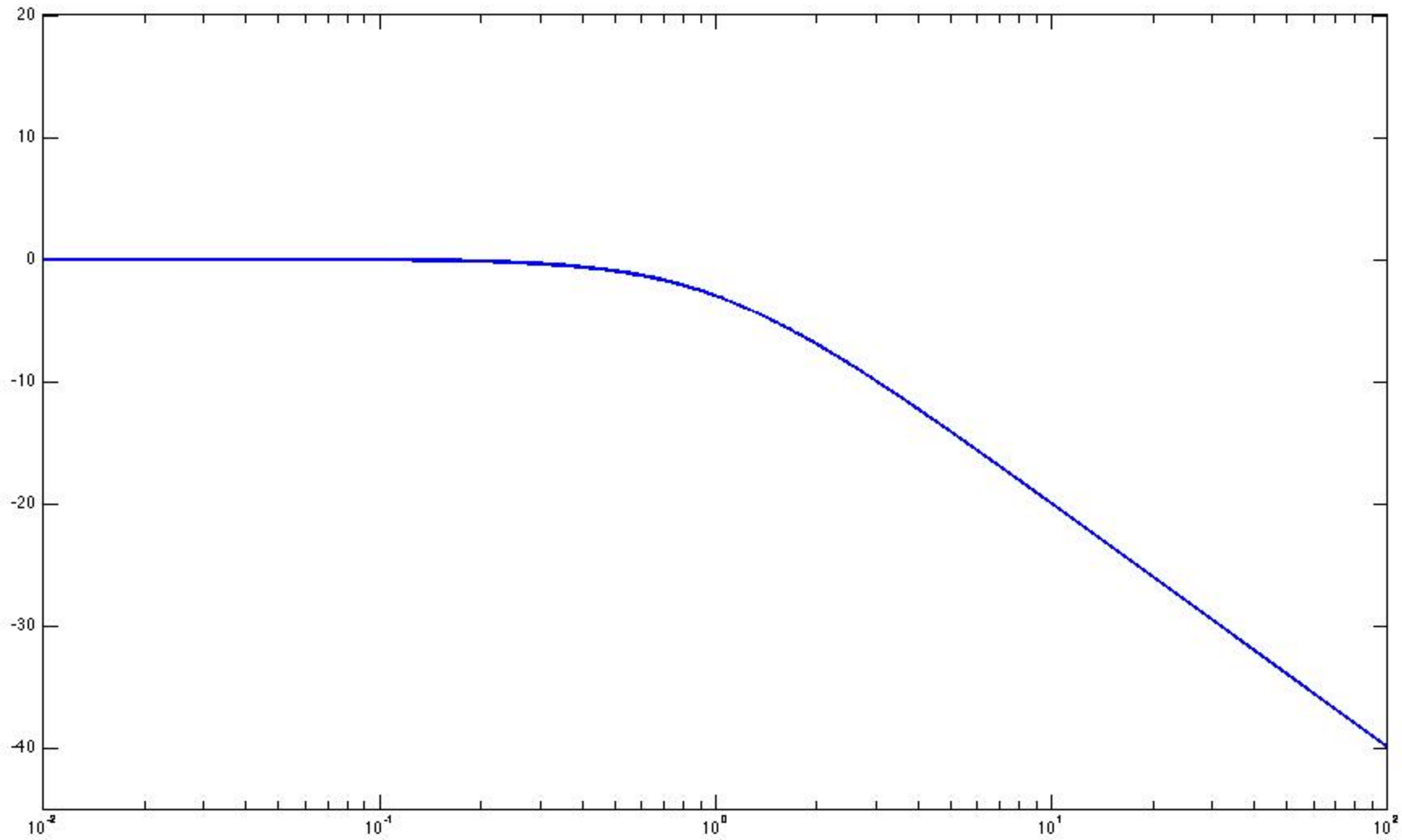
A plot in dB makes it possible to see small values of the FR magnitude $|H(j\omega)|$. This is important when we want to understand the quality of the filter. In particular we can observe the critical values of the frequency for which the character of the filter changes.

Example: Linear plot of the low-pass filter $H(j\omega) = \dfrac{1}{1 + j\omega}$



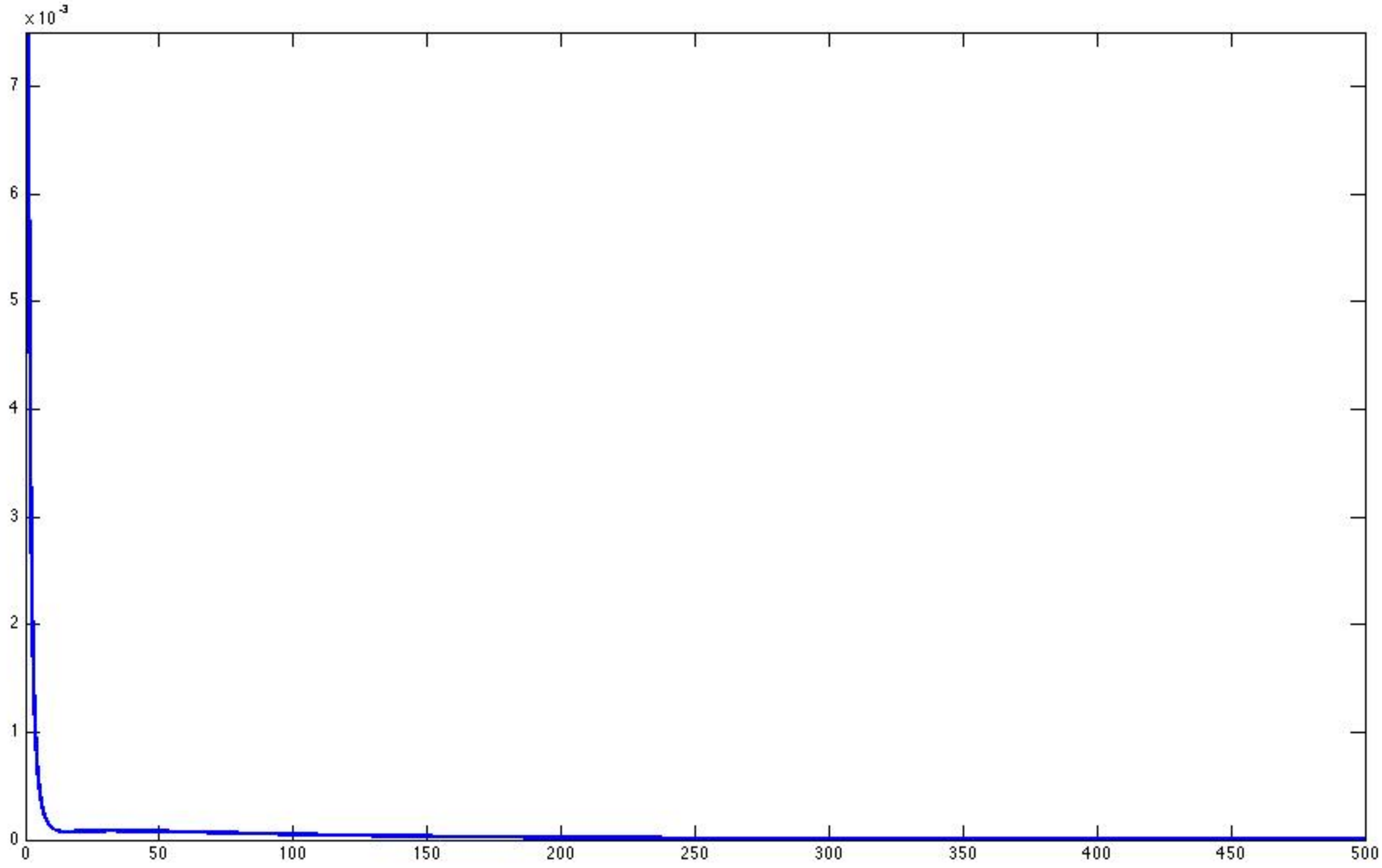
dB representation of Frequency Response

The same function in dB units has a plot:



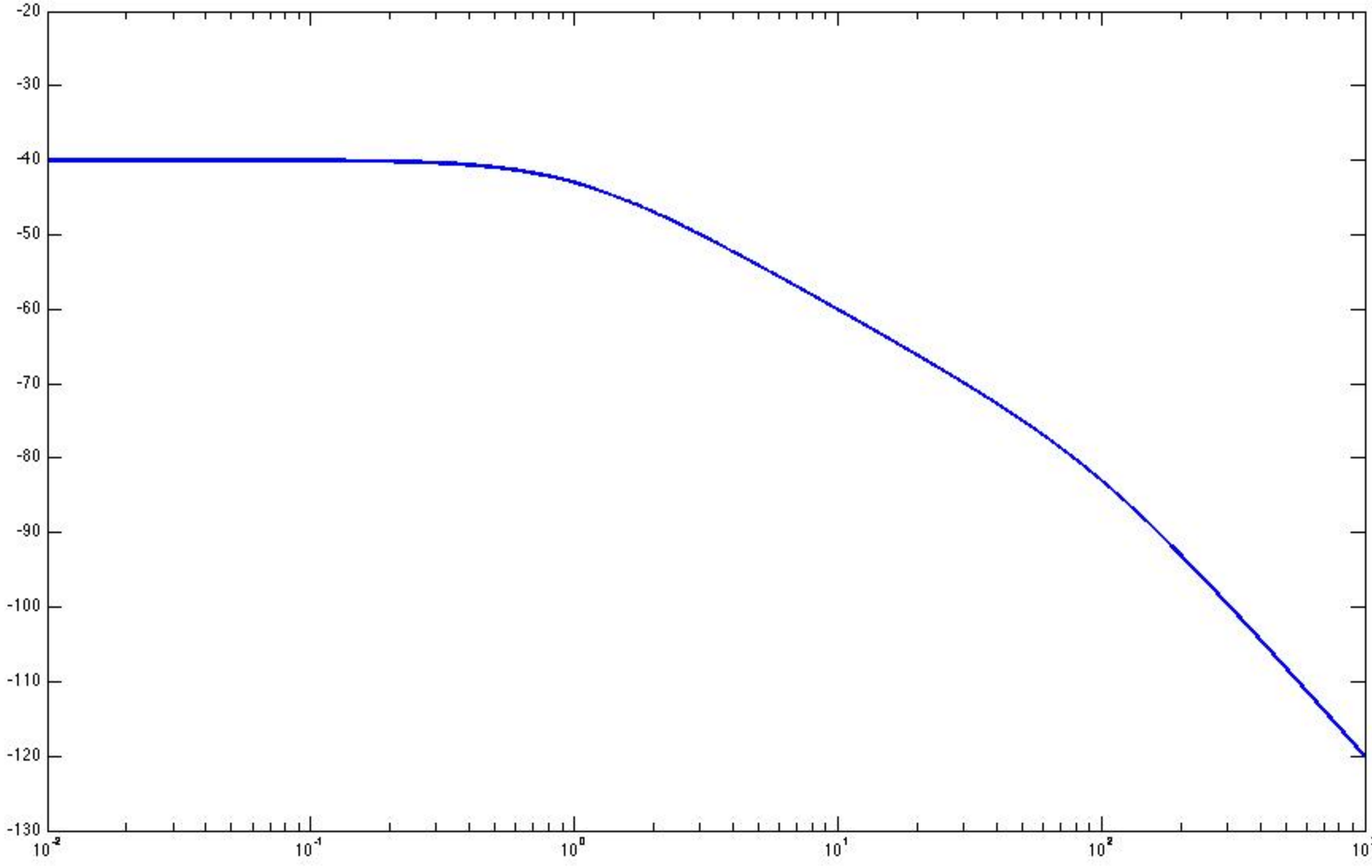
dB representation of Frequency Response

Linear plot of the low-pass filter $H(j\omega) = \dfrac{1}{(1 + j\omega)(100 + j\omega)}$



dB representation of Frequency Response

… and the plot of the same filter in dB:



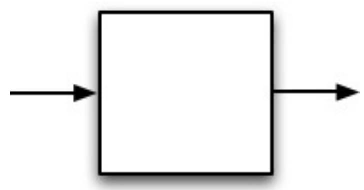
Response of Systems to Co-sinusoids

The FR of an LTI system has the following properties:

$$|H(-j\omega)| = |H(j\omega)| \qquad \angle H(-j\omega) = -\angle H(j\omega)$$

Because of this, the response of the LTI system to real sinusoid or co-sinusoid is as follows:

$$\cos(\omega t + \phi)$$



$$|H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega))$$

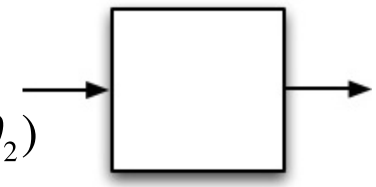
Knowledge of $H(j\omega)$ is enough to know how the system responds to real co-sinusoidal signals

This property is what allows compute the FR experimentally

Response to system to multi-frequency inputs

Use linearity to generalize the response of stable LTI systems to multi-frequency inputs

$$x(t) = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)$$



$$y(t) = A_1 |H(j\omega_1)| \cos(\omega_1 t + \theta_1 + \angle H(j\omega_1)) + A_2 |H(j\omega_2)| \cos(\omega_2 t + \theta_2 + \angle H(j\omega_2))$$

The system acts on each cosine independently

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Signal decompositions in the TD and FD

Shift of perspective:

- In the Time Domain, we see a **signal** as a **function of time**: how strong/weak is at each instant of time
- In the Frequency Domain, we see a **signal** as a **function of frequency**: how strong/weak is at each frequency

A Frequency Domain representation of a signal tells you how much energy of the signal is distributed over each cosine/sine

The different ways of obtaining a FD representation of a signal constitute the **Fourier Transform**, which has 2 categories:

Fourier Series: for periodic signals

Fourier Transform: for aperiodic signals

Each method works for continuous-time and discrete-time signals

Fourier Series of periodic signals

Let $x(t)$ be a "nice" periodic signal with period T_0

Using **Fourier Series** it is possible to write $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k] e^{jk2\pi f_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

The coefficients $X[k]$ are computed from the original signal as

$$X[k] = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

JBJ Fourier

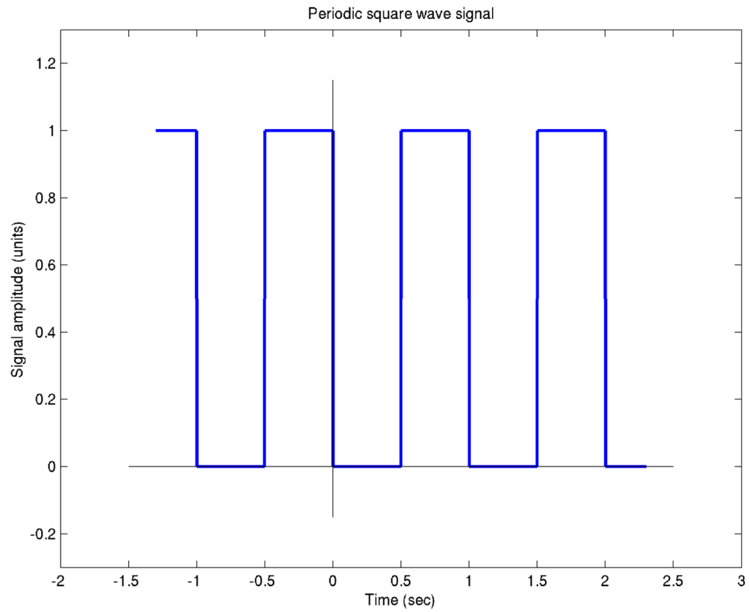
$X[k]$ is the **harmonic function**, and

k is the **harmonic number**

Fourier Example

Unit square wave

$$x(t) = \begin{cases} 0, & 0 < t < 0.5 \\ 1, & 0.5 < t < 1 \end{cases}$$



Compute coefficients

$$X[0] = \frac{1}{1}\int_0^1 x(t)1dt = 0.5$$

$$X[k] = \int_0^1 x(t)e^{-jk2\pi t}dt = \int_{0.5}^1 e^{-jk2\pi t}dt = \frac{1}{-j2k\pi}\left(e^{-jk2\pi} - e^{-jk\pi}\right)$$

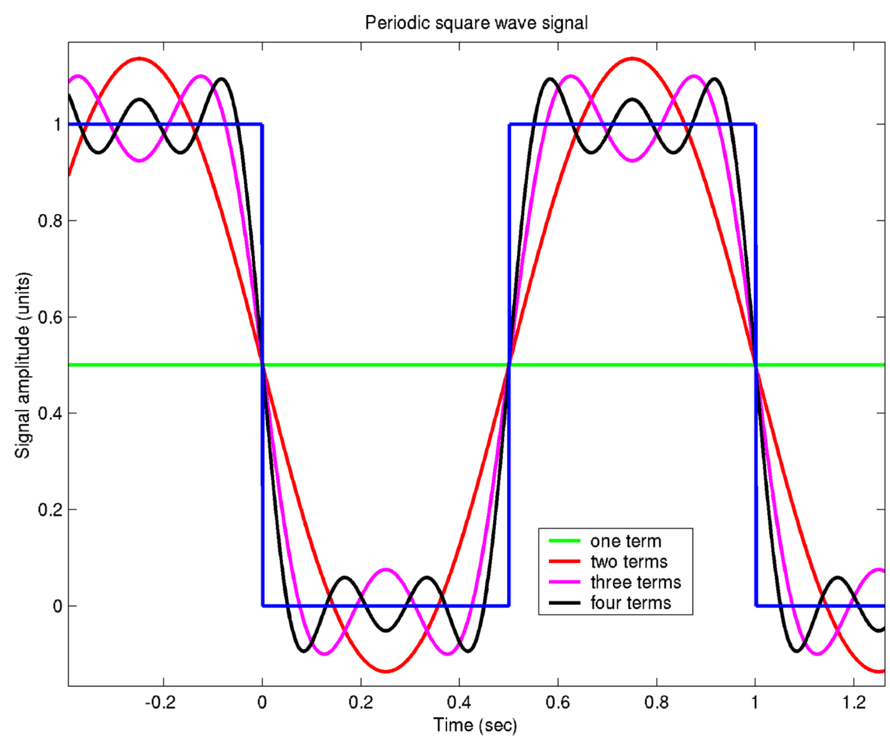
$$= \begin{cases} \dfrac{j}{k\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \qquad k = \ldots,-2,-1,1,2,\ldots$$

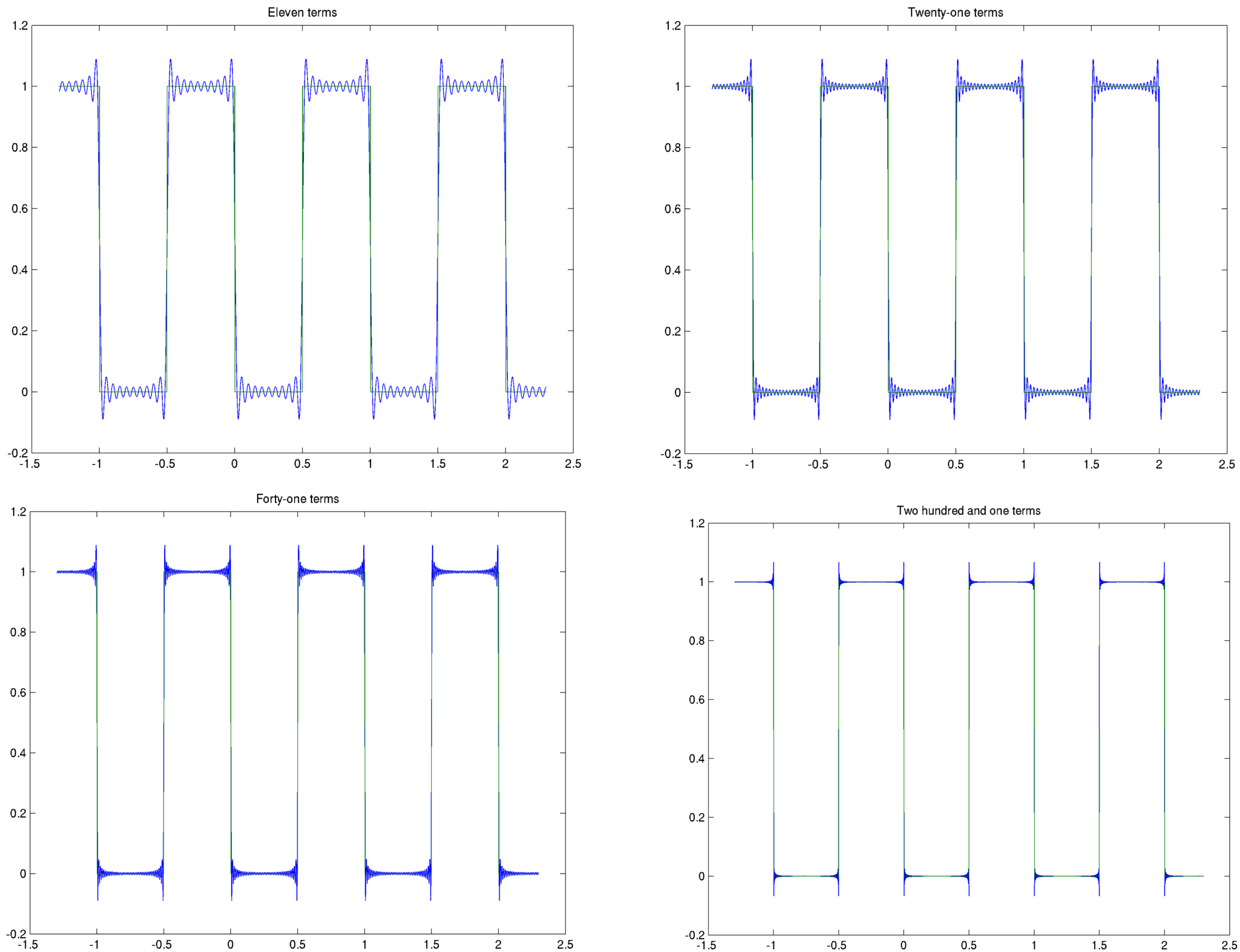
Fourier Example

Square wave

$$x(t) = 0.5 + \sum_{k=1}^{\infty} \frac{j}{(2k-1)\pi} e^{j(2k-1)2\pi t} + \sum_{k=1}^{\infty} \frac{-j}{(2k-1)\pi} e^{-j(2k-1)2\pi t}$$

$$= 0.5 - \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin\left(2\pi(2k-1)t\right)$$




Eleven terms
Twenty-one terms
Forty-one terms
Two hundred and one terms
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

"Proof" of Fourier Series

Fact #1: For any integer i

$$\int_0^{T_0} e^{-ji\omega_0 t}\,dt = T_0 \quad \text{if} \quad i = 0, \quad \text{and} \quad \int_0^{T_0} e^{-ji\omega_0 t}\,dt = 0 \quad \text{if} \quad i \neq 0$$

Proof: If a Fourier Series exists then

$$x(t) = \sum_{i=-\infty}^{\infty} X[i] e^{ji\omega_0 t}$$

Multiply by $e^{-jk\omega_0 t}$ on both sides

$$x(t)e^{-jk\omega_0 t} = \sum_{i=-\infty}^{\infty} X[i] e^{ji\omega_0 t} e^{-jk\omega_0 t} = \sum_{i=-\infty}^{\infty} X[i] e^{j(i-k)\omega_0 t}$$

"Proof" of Fourier Series

Proof (continued):

Integrate over a period on both sides

$$\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \sum_{i=-\infty}^{\infty} X[i]\int_0^{T_0} e^{j(i-k)\omega_0 t}dt$$

and use Fact #1 to show that

$$\sum_{i=-\infty}^{\infty} X[i]\int_0^{T_0} e^{j(i-k)\omega_0 t}dt = T_0 X[k]$$

which leads to the Fourier formula

$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

Graphical description of a periodic signal as a function of frequency

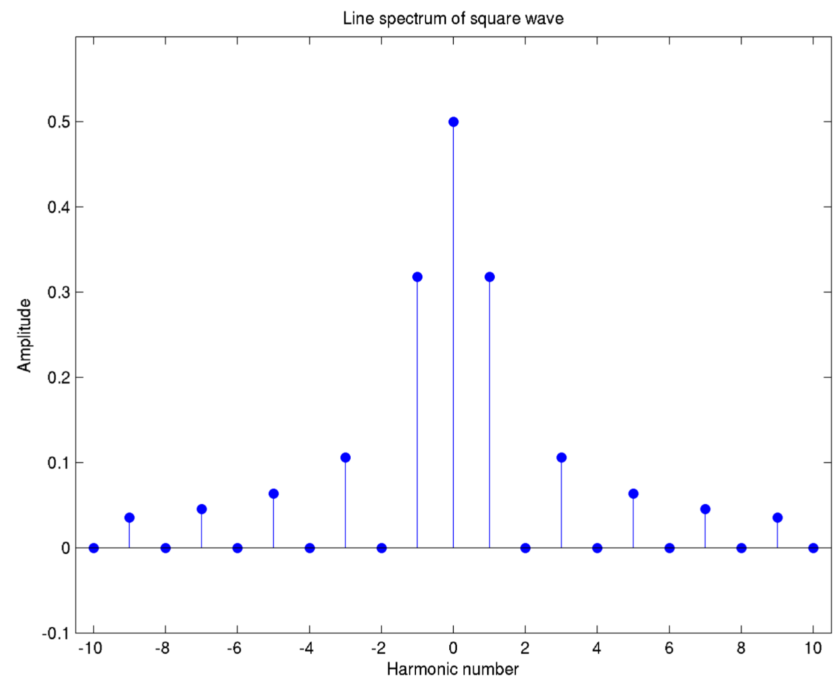
The Frequency Domain graphical representation of a periodic signal is a plot of its Fourier coefficients (FC)

Since the coefficients are complex, the representation consists of:

1) a plot of $|X[k]|$ for different k (the magnitude spectrum)

2) a plot of $\angle X[k]$ for different k (the phase spectrum)

The magnitude spectrum tells us how many frequencies are necessary to obtain a good approximation of the signal



Square wave: most of the signal can be approximated using low frequencies. Discontinuities translate into lots of high frequencies

Properties of the Fourier series

Useful to simplify computation of Fourier coefficients

Linearity: $\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t) \xleftrightarrow{FS} \alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]$

Time shifting: $\mathrm{x}(t-t_0) \xleftrightarrow{FS} e^{-j(k\omega_0)t_0}\,\mathrm{X}[k]$

Frequency shifting: $e^{j(k_0\omega_0)t}\,\mathrm{x}(t) \xleftrightarrow{FS} \mathrm{X}[k-k_0]$

Time reversal: $\mathrm{x}(-t) \xleftrightarrow{FS} \mathrm{X}[-k]$

Multiplication-convolution: $\mathrm{x}(t)\,\mathrm{y}(t) \xleftrightarrow{FS} \mathrm{X}[k]*\mathrm{Y}[k]=\sum_{q=-\infty}^{q=\infty} Y(q)X(k-q)$

Conjugation: $\mathrm{x}^*(t) \xleftrightarrow{FS} \mathrm{X}^*[-k]$

A few more properties in the book…

System response to a periodic signal

To find the response of a (stable) LTI system to a periodic signal of period T_0 :

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$X[k] = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt, \quad k = \ldots,-2,-1,0,1,2,\ldots$$

We use the system FR and change each signal frequency component as follows:

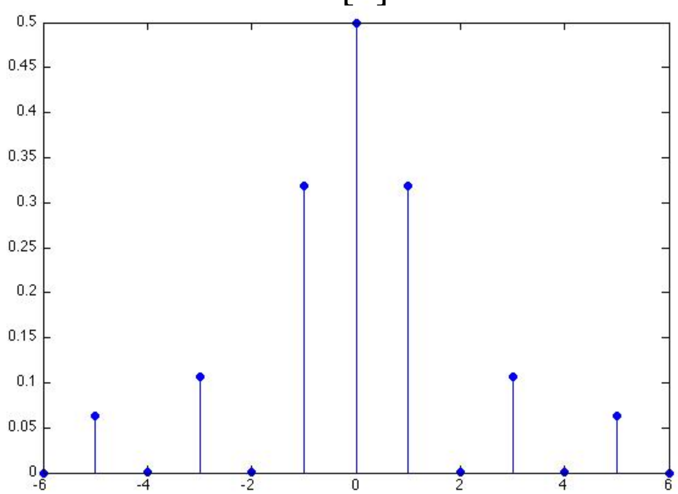
$$y(t) = \sum_{k=-\infty}^{\infty} H(kj\omega_0) X[k] e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} H[k] X[k] e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} X[k] \,|\, H(kj\omega_0) \,|\, e^{j(k\omega_0 t + \angle H(kj\omega_0))} = \sum_{k=-\infty}^{\infty} |\, X[k] \,||\, H[k] \,|\, e^{j(k\omega_0 t + \angle H[k] + \angle X[k])}$$

Graphical signal/systems interaction in the FD

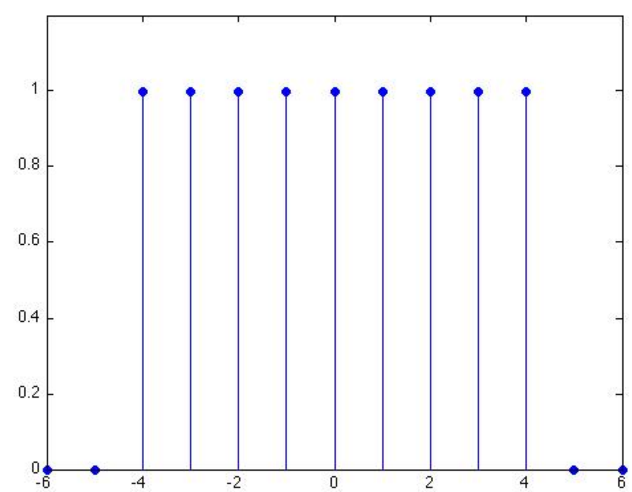
In this way, the interaction of periodic signals and systems in the FD can be seen a simple vector multiplication/vector sum:

$|X[k]|$

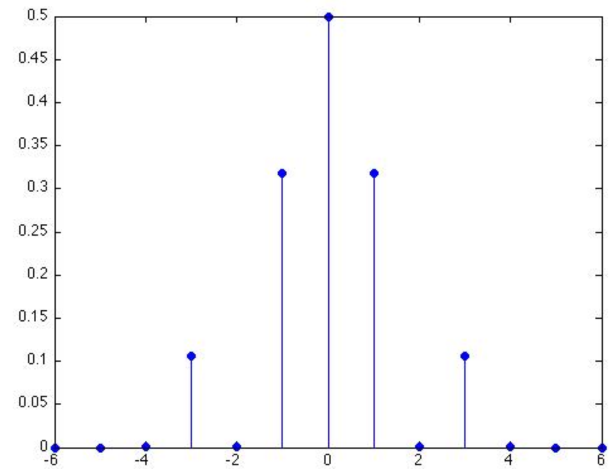


$\times$

$|H[k]|$



$=$



Once we have computed the signal spectrum and system FR, the magnitude spectrum of the output $|Y[k]|$ can be computed in a simple way (compare it with convolution…) The output vector phase corresponds to a vector sum

$$\angle Y[k] = \angle X[k] + \angle H[k]$$

Example

Consider system $y'(t) + y(t) = x(t)$ and consider unit square wave as input $x(t)$. Find the system response $y(t)$

Unit square wave has $T_0 = 1$ and harmonic coefficients

$$X[0] = 0.5 \qquad\qquad X[k] = \begin{cases} \dfrac{j}{k\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

The transfer function of the system is $H(s) = \dfrac{1}{1+s}$. Therefore,

$$H[k] = H(jk2\pi) = \frac{1}{1+2\pi kj} = \frac{1}{\sqrt{1+4\pi^2k^2}} e^{-j\arctan(2\pi k)}$$

The system response is then

$$y(t) = \sum_{k=-\infty}^{\infty} X[k]H[k]e^{j2\pi kt} = 0.5 + \sum_{k=-\infty,\, k \text{ odd}}^{\infty} \frac{j}{k\pi} \frac{1}{\sqrt{1+4\pi^2k^2}} e^{j(2\pi kt - \arctan(2\pi k))}$$

Example-cont'd

Manipulating terms a bit gives the more familiar form

$$y(t) = 0.5 + \sum_{k=-\infty,\ k\ \text{odd}}^{\infty} \frac{j}{k\pi} \frac{1}{\sqrt{1+4\pi^2 k^2}} e^{j(2\pi k t - \arctan(2\pi k))}$$

$$= 0.5 + \sum_{i=1}^{\infty} \frac{j}{(2i-1)\pi} \frac{1}{\sqrt{1+4\pi^2(2i-1)^2}} e^{j(2\pi(2i-1)t - \arctan(2\pi(2i-1)))}$$

$$+ \sum_{i=1}^{\infty} \frac{-j}{(2i-1)\pi} \frac{1}{\sqrt{1+4\pi^2(2i-1)^2}} e^{-j(2\pi(2i-1)t - \arctan(2\pi(2i-1)))}$$

$$= 0.5 - \sum_{i=1}^{\infty} \frac{2}{(2i-1)\pi} \frac{1}{\sqrt{1+4\pi^2(2i-1)^2}} \sin(2\pi(2i-1)t - \text{atan}(2\pi(2i-1)))$$

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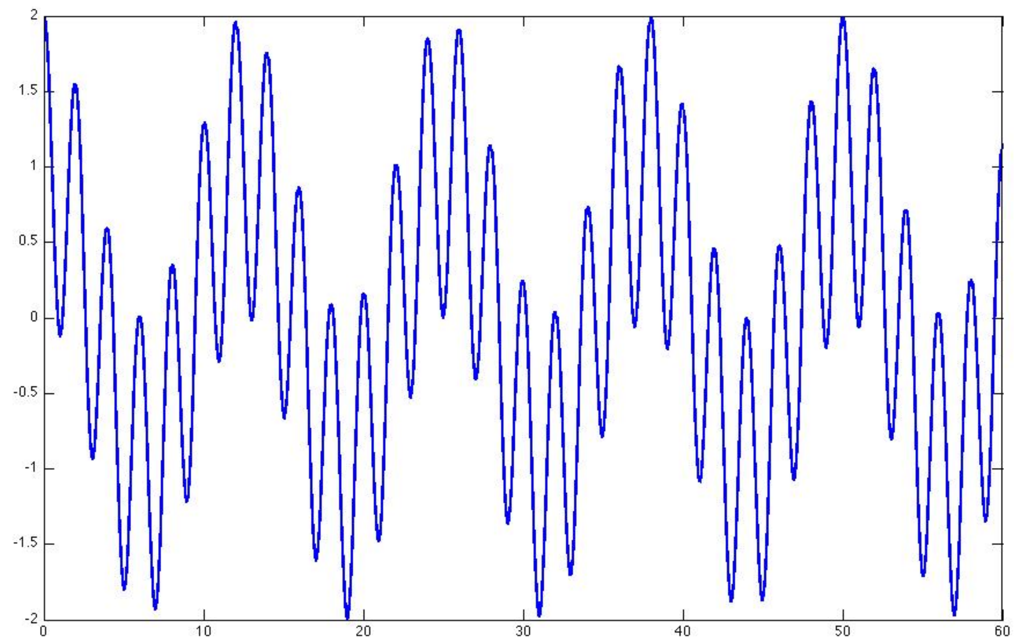
Noise Removal in the Frequency Domain

From our previous discussion on filters/multi-frequency inputs we observe the following: rapid oscillations “on top” of slower ones in signals can be “smoothed out” with low-pass filters

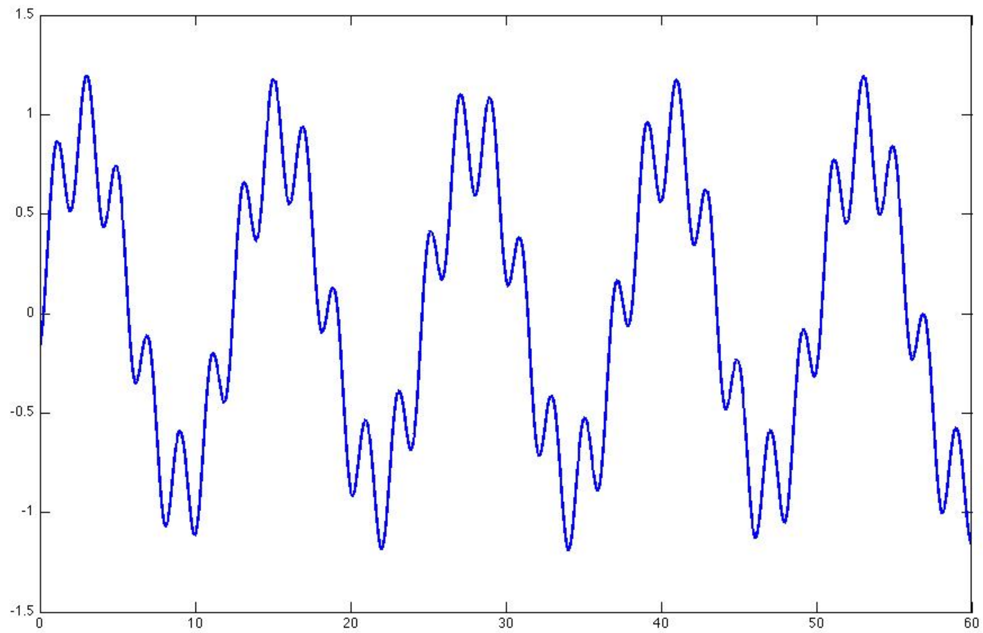
Example: In the signal $x(t) = \cos(t/2) + \cos(\pi t)$ the high frequency component is $\cos(\pi t)$, the low frequency component is $\cos(t/2)$



Noise Removal in the Frequency Domain

The response of the low-pass filter $\frac{1}{1+s}$ to the signal is:

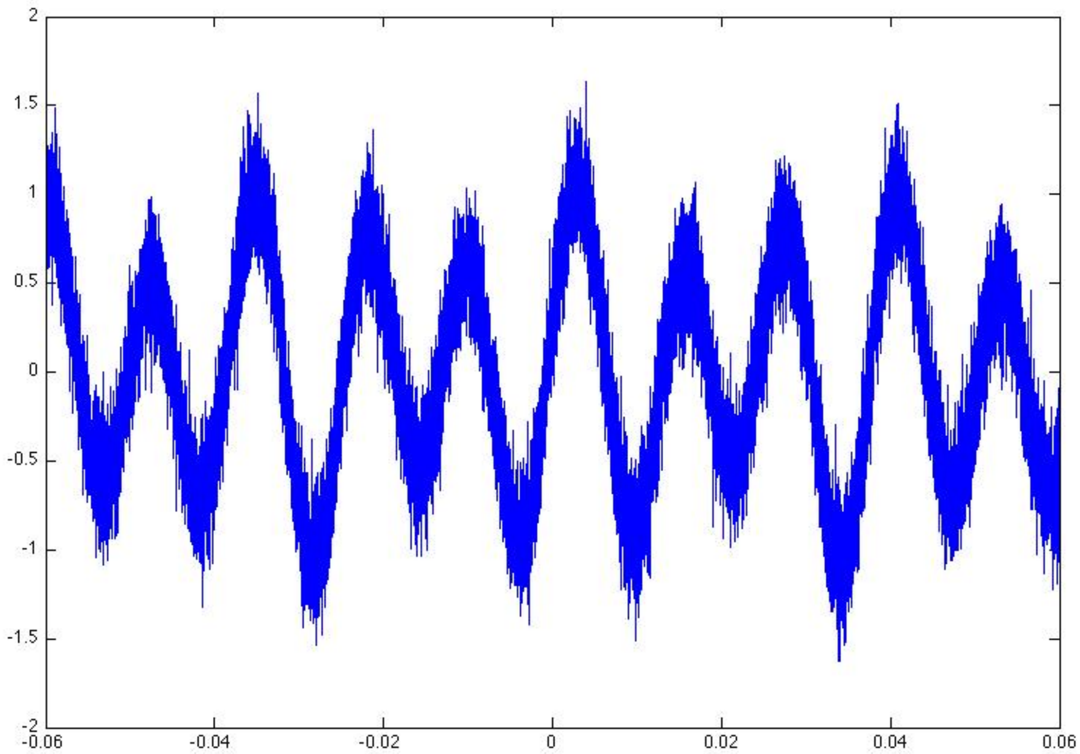
$$y(t) = 0.894\cos(t/2 - 0.46) + 0.303\cos(\pi t - 1.26)$$



The output signal retains the slow oscillation of the input signal while almost removing the high-frequency oscillation

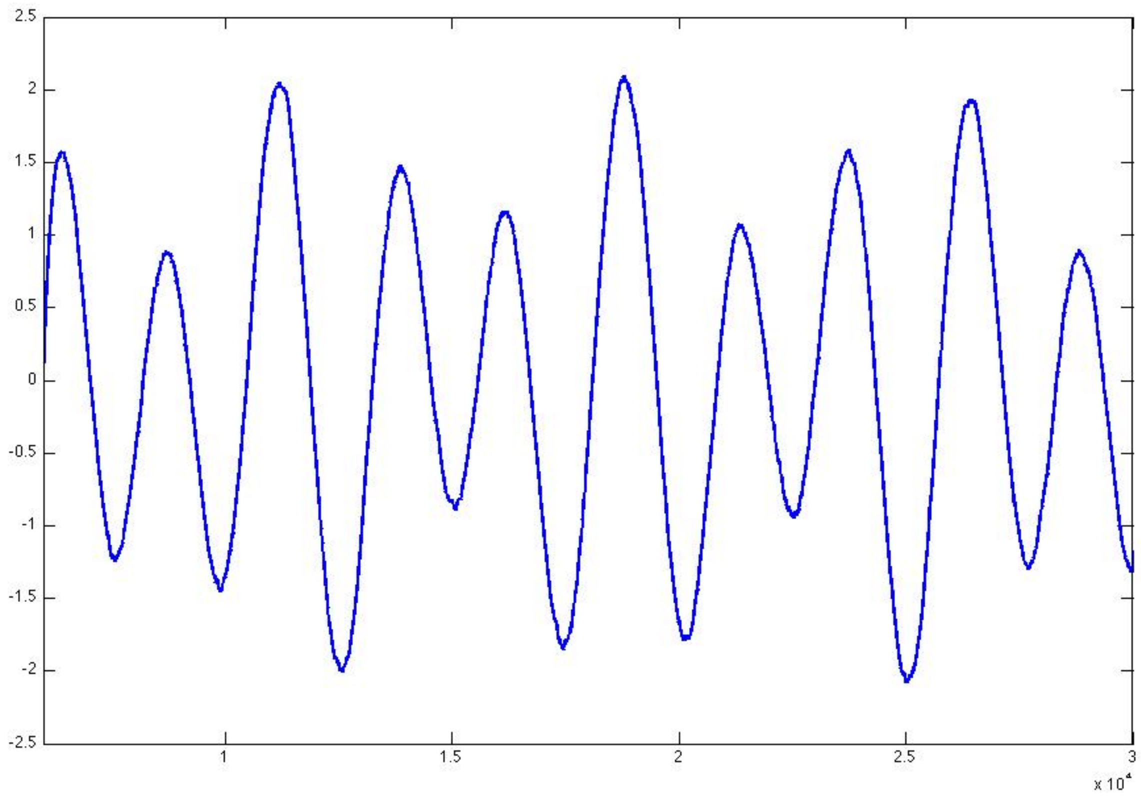
Noise Removal in the Frequency Domain

A low-frequency periodic signal subject to noise can be seen as a low-frequency cosine superimposed with a high-frequency oscillation. For example consider the following noisy signal:



Noise Removal in the Frequency Domain

The steady-state response of a low-pass filter (e.g., the RC low-pass filter) to the signal is the following:



(This has exactly the same shape as the uncorrupted signal, and we are able to remove the noise pretty well)

Noise Removal in the Frequency Domain

The use of **low-pass filters for noise removal** is a **widespread technique**. We now know why this technique works for periodic signals

The same technique works for any signal of finite energy. This is explained through the theory of **Fourier/Laplace transforms**

Depending of the **type of noise and signal, some filters may work better than others.** This leads to the whole are of filter design in signals and systems.

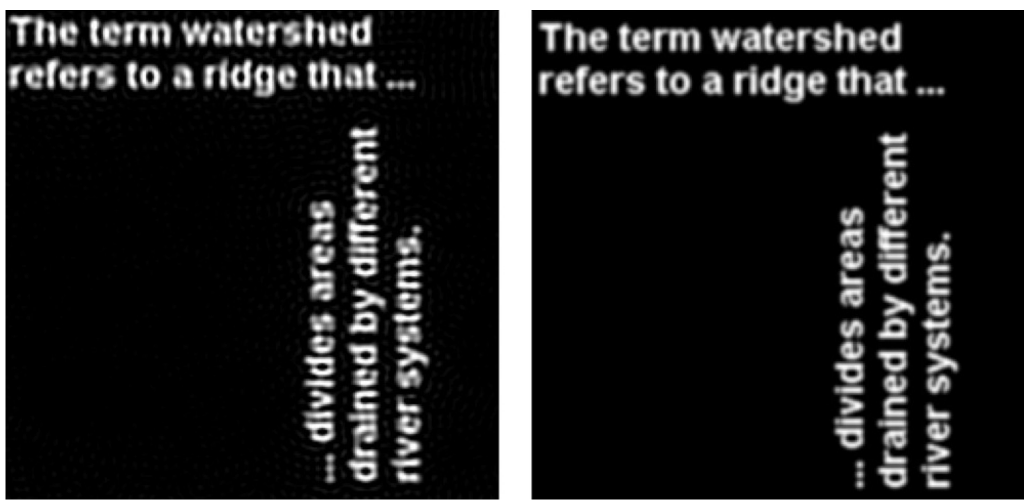
Noise removal in the Frequency Domain

However, in addition to the noise, we may find:

- in audio signals: plenty of **other high-frequency** content
- in image signals: **edges contribute** significantly **to high-frequency** components

Thus an "ideal" low-pass filter tends to blur the data

- E.g., edges in images can become blurred. Observe the output to two different low-pass filters:



This illustrates that one has to be careful in the selection of filter

Summary

Reasons why we study signals in the frequency domain (FD):

(a) Oscillatory inputs and sinusoids are **easier to implement and reproduce** in a lab than impulse signals (which usually we need to approximate) and even unit steps

(b) A pulse-like signal can be expressed as a combination of co-sinusoids, so if we know how to compute the response to a co-sinusoid, then we can know what is the response to a pulse-like signal (and to general periodic signals)

(c) Once signals and systems are in the FD, **computations to obtain outputs are very simple**: we don't need convolution anymore! This is one of the reasons why signal processing is mainly done in the FD. For example, deconvolution (inverting convolutions) is more easily done in the FD

(d) The FD **can be more intuitive** than the TD. For example, how noise removal works is easier to understand in the FD

Summary

Important points to remember:

1. The **output of (stable) LTI systems** to a **complex exponential, sinusoid or co-sinusoid, resp., is again another complex exponential, sinusoid or co-sinusoid, resp.,**

2. These outputs can be computed by knowing the **magnitude and phase of the Frequency Response** *H(jw)* associated with the LTI system

3. **Systems can be classified** according to their Frequency Response **as low-pass, high-pass or band-pass filters**

4. We can **approximate** a wide class of **periodic signals** as a sum of complex exponentials or co-sinusoid **via a Fourier Series (FS) expansion**

5. The **FS expansion** allows us to see signals as **functions of frequency through its (magnitude and phase) spectrum**

6. A (stable) LTI **system acts on each frequency component of the FS of a periodic signal** independently. This leads to fast computations (compare with a convolution…)

7. Low-pass filters are used for **signal "smoothing" and noise removal**