Frequency Response and Continuous-time Fourier Transform

Signals and Systems in the FD-part II

Goals

I. (Finite-energy) signals in the Frequency Domain

- The Fourier Transform of a signal
- Classification of signals according to their spectrum (low-pass, high-pass, band-pass signals)
- Fourier Transform properties

II. LTI systems in the Frequency Domain

- Impulse Response and Frequency Response relation
- Computation of general system responses in the FD

III. Applications to audio signals

- A simple design of an equalizer

Objectives

How can we extend the Fourier Series method to other signals?

There are two main approaches:

The Fourier Transform (used in signal processing)

The Laplace Transform (used in linear control systems)

The Fourier Transform is a particular case of the Laplace Transform, so the properties of Laplace transforms are inherited by Fourier transforms. One can compute Fourier transforms in the same way as Laplace transforms.

Fourier Transform

Generalization of Fourier Series to aperiodic functions

Recall, for a periodic function $x(t)$ of period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \qquad X[k] = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jk\omega_0 t} dt$$

For an aperiodic function, take $T_0 \to \infty$

$$\underbrace{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega}_{\text{Inverse Fourier transform}} \qquad \underbrace{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{\text{Fourier transform}} \qquad \left(\lim_{T_0 \to \infty} \frac{1}{T_0} \approx \frac{d\omega}{2\pi} \right)$$

Requires $x(t)$ to be absolutely integrable $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Relationship with Laplace transform

The Fourier Transform is a particular case of Laplace transform

$$L(s) = \int_{-\infty}^{\infty} g(t) e^{-st} dt \qquad F(j\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \qquad F(j\omega) = L(j\omega)$$

$F(j\omega)$ is often called spectrum or amplitude spectral density (spectral refers to 'variation with respect to frequency', density refers to 'amplitude per unit frequency')

Sometimes, the transform is seen as a function of cyclic frequency $2\pi f = \omega$. That is,

$$F(j\omega) = F(j2\pi f) = \overline{F}(f)$$

$$F(j\omega) = \overline{F}(j\omega/2\pi) = \overline{F}(f)$$

Transform methods get more powerful

Fourier series to Fourier transform to Laplace transform

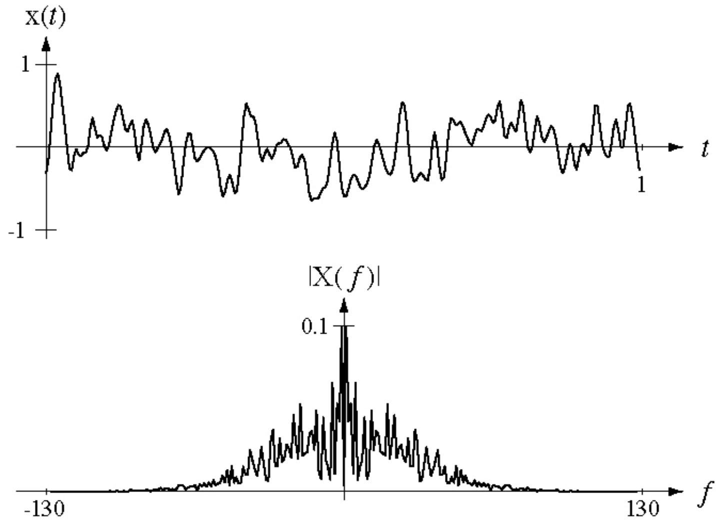
A finite-amplitude, real signal can be represented as

- periodic case: countable infinity of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

- aperiodic case: uncountable infinity of complex exponentials

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$



Important features may be obscure in TD, but crystalline in FD.

General signals in the Frequency Domain

The **signal representation in the Frequency Domain** is the graph of $|F(j\omega)|$ and of $\angle F(j\omega) = \arctan(\mathrm{Im}(F(j\omega)/\mathrm{Re}(F(j\omega))$

One can classify signals into:

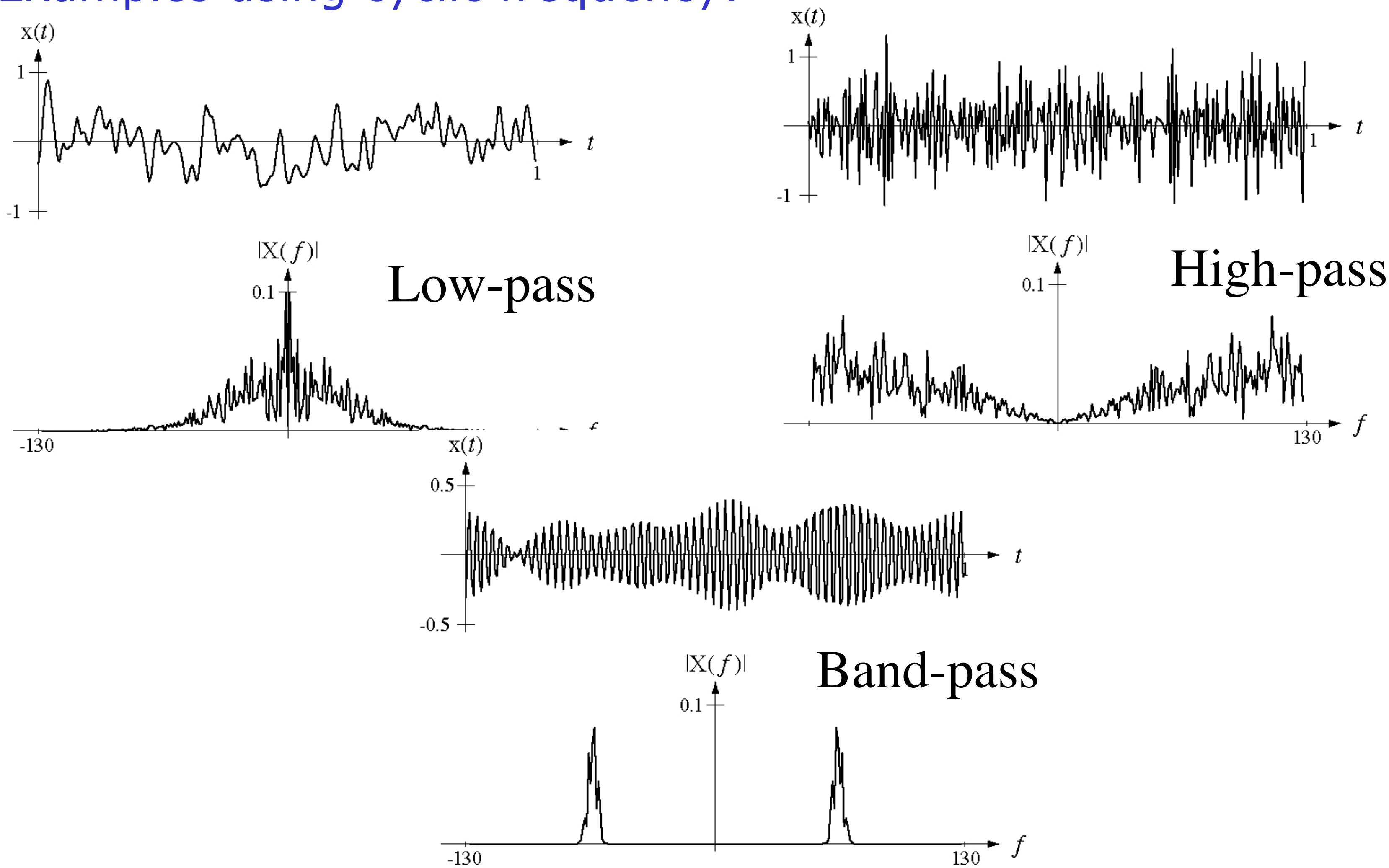
Low-pass signal: The energy of the signal is **concentrated at low frequencies** (these signals have "slow transitions" in the time domain and are smooth)

High-pass signal: The energy of the signal is **concentrated at high frequencies** (these signals have zero mean and rapidly changing values; e.g. noise)

Band-pass signal: The spectrum vanishes at low and high frequencies and **concentrates in an intermediate frequency band** (signals that look more like a pure sinusoid with high frequency)

Graphical Interpretation TD versus FD

Examples using cyclic frequency:



Examples: computation of FT

Let’s compute the FT of $x(t) = e^{-a|t|}, \ \mathrm{Re}(a) > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at-j\omega t}dt + \int_{0}^{\infty} e^{-at-j\omega t}dt$$

$$= \left(\frac{1}{a-j\omega}e^{(a-j\omega)t}\right)\Bigg|_{-\infty}^{0} - \left(\frac{1}{a+j\omega}e^{-(a+j\omega)t}\right)\Bigg|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

Let’s compute the FT of $x(t) = 1$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-j\omega t}dt = ?$$

Integral does not converge

Computation of FT

What do we do? Calculate FT of $x(t)e^{-\sigma t}$, $\sigma > 0$ and then take $\sigma \to 0$

$$X_\sigma(j\omega) = \frac{2\sigma}{\sigma^2 + \omega^2}$$

Now,

$$\lim_{\sigma \to 0} X_\sigma(j\omega) = 0, \text{ if } \omega \neq 0 \qquad \lim_{\sigma \to 0} X_\sigma(j\omega) = \infty, \text{ if } \omega = 0 \qquad \int_{-\infty}^{\infty} X_\sigma(j\omega) d\omega = 2\pi$$

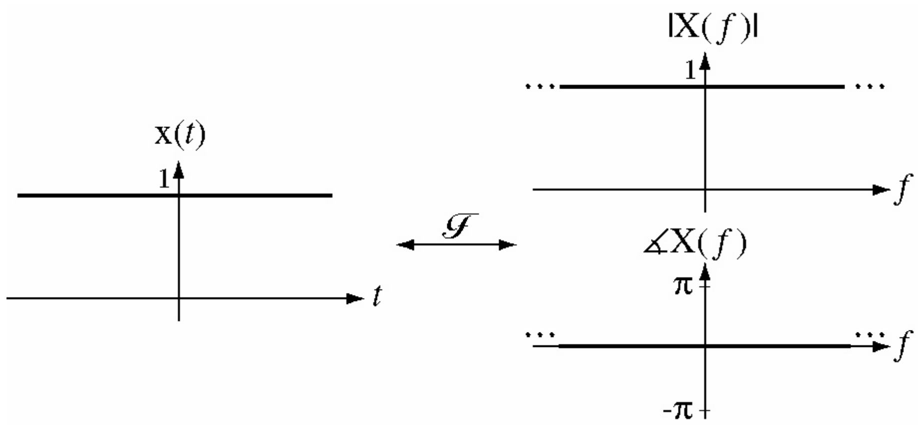
Therefore, $X(j\omega) = 2\pi\delta(\omega)$

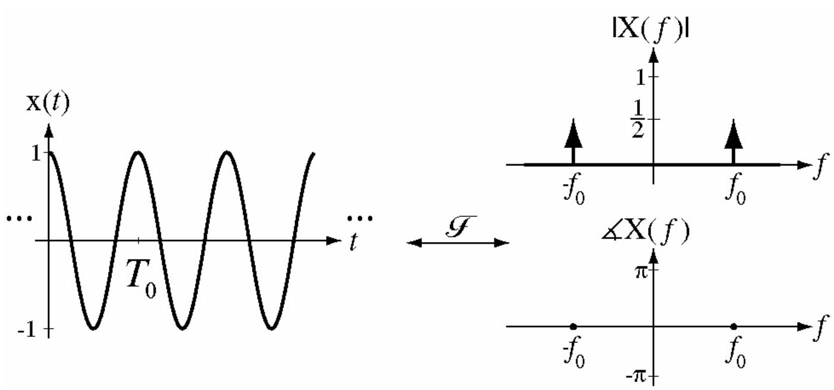
Similarly, one can also compute

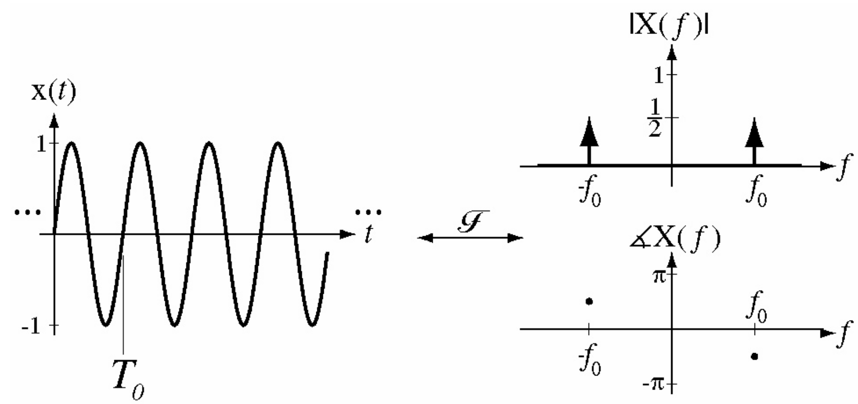
$$\cos(at) \leftrightarrow \pi(\delta(\omega - a) + \delta(\omega + a))$$

$$\sin(at) \leftrightarrow j\pi(\delta(\omega + a) - \delta(\omega - a))$$

Graphical representation







Observe this is the same as the Fourier Series spectrum graphs!

TD and FD representation of a signal

The Time Domain and Frequency Domain representations are used to **emphasize different aspects**

TD graph: you can see when something occurs and the amplitude of occurrence. For example, this is good for describing step or discontinuous signals

FD graph: the magnitude and phase gives you information about what produces the oscillations present in the signal. Good for describing audio signals

Example: sound from ringing a wine glass. The harmonics, fundamental frequency, tells us the nature of the material that produced it. This information is not contained in a single sample time of a TD description

TD versus FD: what to choose?

In general, the use of TD or FD signal representations depends on the application you are working with

For ECG monitors, you want to emphasize time occurrences, and produce filters that generate good step responses*. So you study time domain representations of step responses

For a hearing aid, you want to emphasize certain frequency ranges, so you want to optimize your filter with respect to frequency responses. So you study frequency domain representations of audio signals

Filters that perform well in terms of frequencies do not perform well in terms of step responses, and the other way around

*a good step response should have a shape as close to a step as possible

Fourier Transform pairs Table

A few more examples of Fourier Transform pairs from the book (Table 10.1, **observe there are two columns**, one for f and one for ω)

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{j\omega + a}$$

$$\cos(at) \leftrightarrow \pi(\delta(\omega - a) + \delta(\omega + a))$$

$$\sin(at) \leftrightarrow j\pi(\delta(\omega + a) - \delta(\omega - a))$$

$$rect(t) \leftrightarrow \sin c(\omega / 2\pi)$$

$$\sin c(t) \leftrightarrow rect(\omega / 2\pi)$$

Fourier Transform properties (radian freq)

Linearity	$ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)$
Shift in time	$x(t-c) \leftrightarrow X(j\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{\lvert a \rvert} X(j\frac{\omega}{a})$
Frequency scaling	$\frac{1}{\lvert a \rvert} x(\frac{t}{a}) \leftrightarrow X(aj\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega-\omega_0))$
Modulation	$x(t)\sin\omega_0 t \leftrightarrow \frac{j}{2}\left[X(j\omega+j\omega_0)-X(j\omega-j\omega_0)\right]$
	$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}\left[X(j\omega+j\omega_0)+X(j\omega-j\omega_0)\right]$
Differentiation	$\frac{d^n x(t)}{dt^n} \leftrightarrow \left(j\omega\right)^n X(j\omega)$
Integration	$\int_{-\infty}^{t} x\left(\lambda\right)d\lambda \leftrightarrow \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(j\omega)$

Example: calculate FT of a periodic funtion

Let $x(t)$ be periodic of period T_0

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

Using linearity and frequency shifting,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_0)$$

In other words,

the Fourier transform is a generalization of the Fourier series

Convolution property of Fourier transforms

Convolution/Multiplication duality:

$$h(t) * f(t) = \int_{-\infty}^{\infty} h(t-\sigma) f(\sigma) d\sigma \leftrightarrow H(j\omega) F(j\omega)$$

$$h(t) f(t) \leftrightarrow \frac{1}{2\pi} H(j\omega) * F(j\omega)$$

In particular we have

$$\boxed{h(t) \overset{F}{\longleftrightarrow} H(j\omega)}$$

What are the implications?

For a stable linear system we can compute system responses as

$$Y\left(j\omega\right) = H\left(j\omega\right)X\left(j\omega\right)$$

$Y\left(j\omega\right)$, $X\left(j\omega\right)$ are signal Fourier transforms

$H\left(j\omega\right)$ is the system frequency response

Spectra $\left|Y(j\omega)\right| = \left|H(j\omega)\right|\left|X(j\omega)\right|, \quad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$

The Frequency Response contains all the information needed to determine any system response

Deconvolutions can be computed by division in the FD:

If $y(t) = h(t) * x(t)$, then

$$x(t) = F^{-1}\{Y(j\omega)/H(j\omega)\}$$

(can be done for invertible systems)

Signals/systems in the FD

Similarly to the Fourier Series case, **once the input signal and system Fourier transforms are computed, the response can be obtained through simple multiplication and sum operations**

Additionally, the FT/IFT can be approximated in a computer through special routines: the FFT (Fast Fourier Transform) and the IFFT (Inverse Fast Fourier Transform)

Overall, this makes **the whole process in the FD much faster than convolution in the TD**. The only disadvantage is that the FT only applies to finite energy signals

Signals/systems in the FD

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III. <u>Applications to audio signals</u>

- <u>A simple design of an equalizer</u>

A simple design of an equalizer

Applications of the Fourier Transform theory are many. One of them is the design of equalizers to accomplish different effects on audio signals:

-**Bass volume** or low frequencies in your audio system can be implemented using a low-pass filter;

$$H_1(\omega) = \frac{\omega_c}{j\omega + \omega_c}$$

-**Treble volume** or high frequencies can be implemented using a high-pass filter

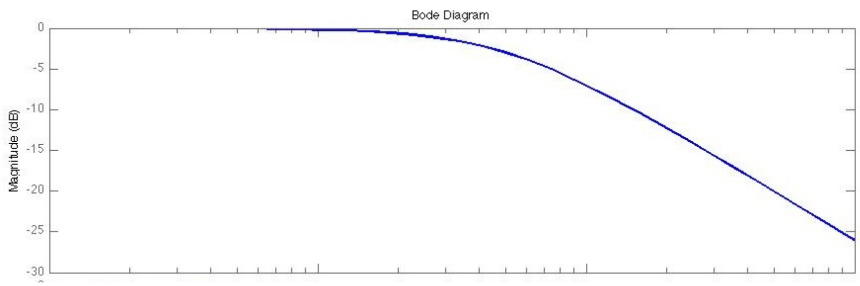
$$H_2(\omega) = \frac{j\omega}{j\omega + \omega_c}$$

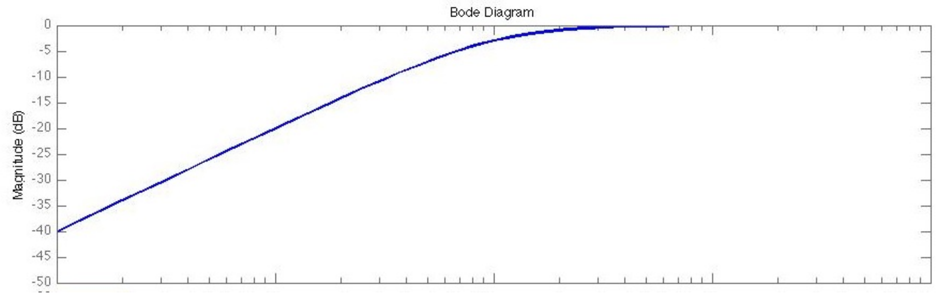
-The previous two filters **can be combined together** to filter only those sounds with frequencies around a value of ω_c

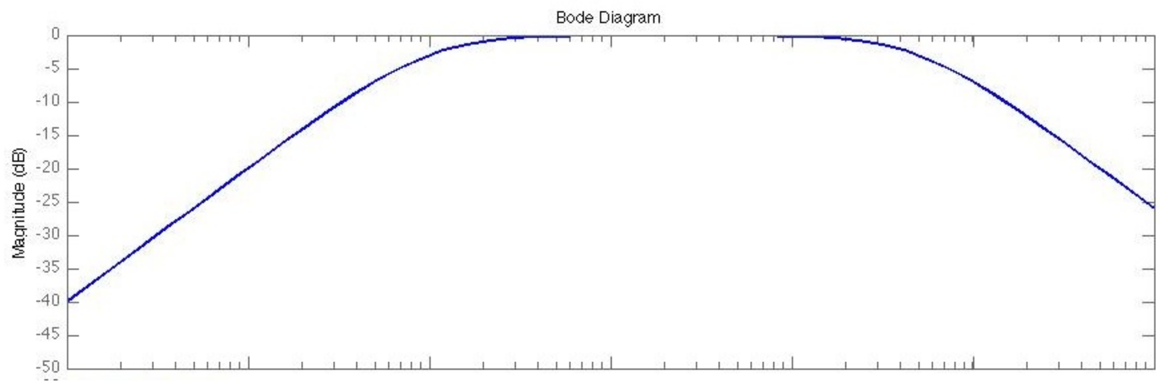
$$H_3(\omega) = \frac{\omega_{c1}}{j\omega + \omega_{c1}} \times \frac{j\omega}{j\omega + \omega_{c2}}$$

A simple design of an equalizer

Example graphs of these filters for a given values of the cutoff frequencies $\omega_{c-lowpass} = 50000,\ \omega_{c-highpass} = 100$:

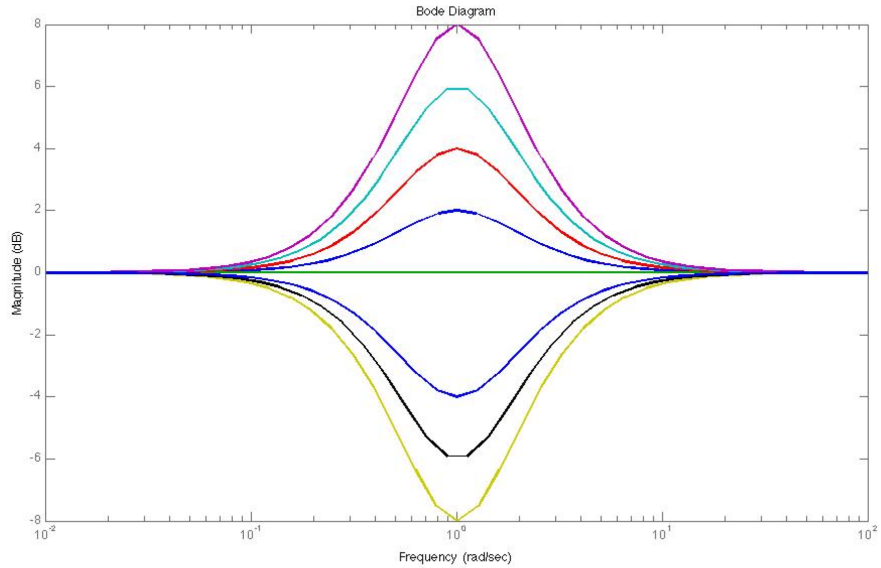






A simple design of an equalizer

Another example of band-pass filter centered at a given frequency $\omega_0 = 1$ and whose magnitude can be adjusted is the following:

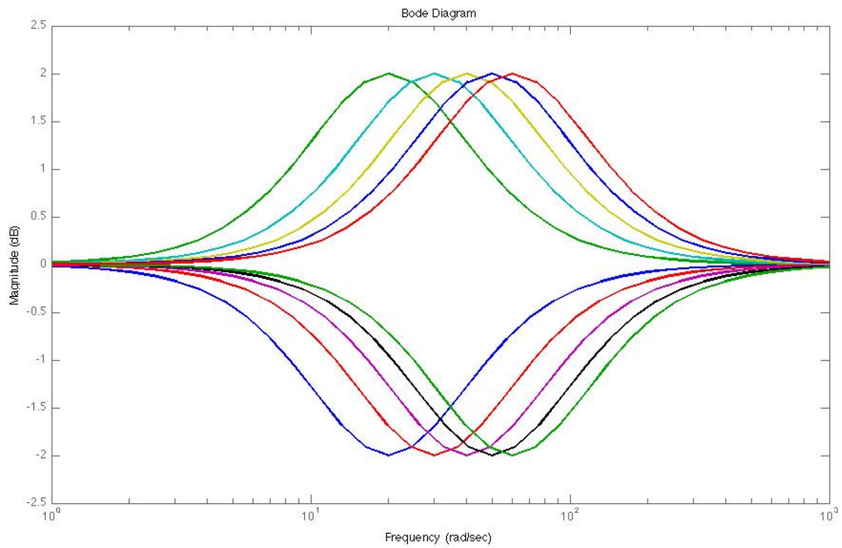


(Each color corresponds to a different choice of β)

$$H_1(j\omega) = \frac{(j\omega)^2 + j2\omega_0\omega/10^{\beta} + \omega_0{}^2}{(j\omega)^2 + j2\omega_0\omega \times 10^{\beta} + \omega_0{}^2}$$

A simple design of an equalizer

A simple equalizer can be built by “connecting in series” band-pass filters like the previous one as follows:



The center frequencies of each band-pass filter are 20Hz, 30Hz, 40Hz,…. The Frequency Response of the equalizer is obtained by summing all the band-pass filters’ FR. By choosing different values of the parameters beta, one can emphasize or de-emphasize any frequency range in an audio signal

Summary

Important points to remember:

A signal can be classified into a **low-pass, high-pass or band-pass signal** depending on its magnitude and phase spectrum

In the time domain**, low-pass signals** correspond to signals with **slow transitions**. **High-pass signals** correspond to signals with **fast transitions**. **Band-pass signals** look like **sinusoids/co-sinusoids.**

The Fourier Transform theory allows us to **extend the techniques and advantages of Fourier Series** to more general signals and systems

In particular **we can compute the response of a system to a signal by multiplying the system Frequency Response and the signal Fourier Transform**. (And we can avoid convolution)

The **Fourier Transform of** the **Impulse Response** of a system **is** precisely **the Frequency Response**

The **Fourier Transform theory can be used to** accomplish different audio effects, e.g. the **design** of **equalizers**