MAE 286: Hybrid Systems (W14)

Homework #6

Due on 2/25/14

1. Consider the hybrid system with data $C = \mathbb{R}$, $D = \emptyset$,

$$F(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$$

   and any G you want. Does this system satisfy the basic assumptions? Is it well-posed?

2. Construct an example of a hybrid automaton that has unstable dynamics in each of its 2 discrete modes, but for which the equilibrium $x_e = 0$ is globally stable. Plot a phase portrait of your system.

3. Consider the hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^2$ and data

$$\begin{aligned} C &:= \left\{x \in \mathbb{R}^2 \mid \|x\| < 1\right\}, & f(x) &:= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \\ D &:= \left\{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \in (-\tfrac{1}{2}, 0]\right\}, & g(x) &:= \tfrac{1}{2}x \end{aligned}$$

   Answer the following questions, properly justifying your response in each case:

   - Is the set $A = \{(0,0)\}$ uniformly globally stable? Is A UGpAS?
   - Is the set $A = \left\{x \in \mathbb{R}^2 \mid \|x\| = \frac{1}{2}\right\}$ uniformly globally stable? Is A UGpAS?

4. This exercise introduces you to the Hybrid Equations (HyEq) Matlab Toolbox toolbox developed for the modeling framework studied in class. The code and instructions are available at `http://www.mathworks.com/matlabcentral/fileexchange/41372-hybrid-equations-toolbox-v2-0`

   You can get familiarized with the toolbox by watching a 30-min webinar at `https://www.mathworks.com/videos/hyeq-a-toolbox-for-simulation-of-hybrid-dynamical-systems-81992.html`

   Once you are familiar with it, let us use it! Hysteresis is a key player in genetic regulatory networks. Consider the following descriptive model of a genetic regulatory network in the mammalian sclera. The state is

$$z = (x_1, x_2, x_3, q_1, q_2, q_3, q_4) \in Z = \mathbb{R}^3_{\geq 0} \times \{0,1\}^4$$

   Here, x_1 represents the protein concentration of TIMP-2 (Type II tissue inhibitor of the matrix metalloproteinases), x_2 the concentration of MT1-MMP (membrane-type I matrix metalloproteinase), and x_3 the concentration of MMP-2 (Type II matrix metalloproteinase). The discrete states q_1, q_2, q_3, q_4 define different modes depending on whether protein concentrations have reached large enough values to express or inhibit other proteins.

   For the continuous flow, positive constants γ_1, γ_2, γ_3 define the decay rates and k_1, k_2, k_3 define the growth rates, respectively, for each of the concentrations. For the jumps, hysteresis functions are associated with the thresholds θ_1 (TIMP-2 level for MT1-MMP expression), θ_2 (MT1-MMP level for MMP-2 expression), θ_3 (TIMP-2 level for MT1-MMP/MMP-2 inhibition), and θ_4 (4 MMP-2 level for TIMP-2 expression), and a hysteresis half-width constant h_i is associated with each threshold.

In this way, the flow map is given by

$$F(z) = \begin{pmatrix} k_1 q_4 - \gamma_1 x_1 \\ k_2 q_1 (1 - q_3) - \gamma_2 x_2 \\ k_3 q_2 (1 - q_3) - \gamma_3 x_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The jump set is $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned} D_1 &= \{z \in Z \mid q_1 = 1,\ x_1 \leq \theta_1 - h_1\} \cup \{z \in Z \mid q_1 = 0,\ x_1 \geq \theta_1 + h_1\} \\ D_2 &= \{z \in Z \mid q_2 = 1,\ x_2 \leq \theta_2 - h_2\} \cup \{z \in Z \mid q_2 = 0,\ x_2 \geq \theta_2 + h_2\} \\ D_3 &= \{z \in Z \mid q_3 = 1,\ x_1 \leq \theta_3 - h_3\} \cup \{z \in Z \mid q_3 = 0,\ x_1 \geq \theta_3 + h_3\} \\ D_4 &= \{z \in Z \mid q_4 = 1,\ x_3 \leq \theta_4 - h_4\} \cup \{z \in Z \mid q_4 = 0,\ x_3 \geq \theta_4 + h_4\} \end{aligned}$$

and the flow set is $C = \overline{Z \setminus D}$. Finally, the jump map is

$$G(z) = \begin{cases} g_1(z) & z \in D_1 \setminus (D_2 \cup D_3 \cup D_4) \\ g_2(z) & z \in D_2 \setminus (D_1 \cup D_3 \cup D_4) \\ g_3(z) & z \in D_3 \setminus (D_1 \cup D_2 \cup D_4) \\ g_4(z) & z \in D_4 \setminus (D_1 \cup D_2 \cup D_3) \\ \{g_1(z), g_2(z), g_3(z), g_4(z)\} & z \in D_1 \cap D_2 \cap D_3 \cap D_4, \end{cases}$$

where,

$$\begin{aligned} g_1(z) &= (x_1, x_2, x_3, 1 - q_1, q_2, q_3, q_4) \\ g_2(z) &= (x_1, x_2, x_3, q_1, 1 - q_2, q_3, q_4) \\ g_3(z) &= (x_1, x_2, x_3, q_1, q_2, 1 - q_3, q_4) \\ g_4(z) &= (x_1, x_2, x_3, q_1, q_2, q_3, 1 - q_4). \end{aligned}$$

Let the thresholds be given by $\theta_1 = 0.4$, $\theta_2 = 0.5$, $\theta_3 = 0.6$, $\theta_4 = 0.7$, with associated half-width constants $h_j = 0.01$, $j \in \{1, 2, 3, 4\}$. Also, let the decay rates be $\gamma_i = 1$, $i \in \{1, 2, 3\}$. Simulate the following scenarios:

(i) $k_1 = 1$, $i \in \{1, 2, 3\}$; and initial condition $x_1(0) = 0.15$, $x_2(0) = 0.45$, $x_3(0) = 0.8$, $q_1(0) = 1$, $q_2(0) = 1$, $q_3(0) = 0$, $q_4(0) = 1$.

(ii) $k_1 = .55$, $k_2 = 1$, $k_3 = 0.9$; and initial condition $x_1(0) = 0.45$, $x_2(0) = 0.6$, $x_3(0) = 0.8$, $q_1(0) = 1$, $q_2(0) = 1$, $q_3(0) = 0$, $q_4(0) = 1$.

(iii) $k_1 = 1$, $i \in \{1, 2, 3\}$; and initial condition $x_1(0) = 0.45$, $x_2(0) = 0.45$, $x_3(0) = 0.8$, $q_1(0) = 1$, $q_2(0) = 1$, $q_3(0) = 0$, $q_4(0) = 1$.

For each scenario, describe what is the limiting behavior that you observe.