MAE 286: Hybrid Systems (W14)
Homework #7

Due on 3/4/14

1. Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$
\begin{aligned}
C &:= \{x : x_1 \geq 0\}, & f(x) &:= \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} x \qquad \forall x \in C, \\
D &:= \{x : x_1 = 0, x_2 \leq 0\}, & g(x) &:= -\gamma x \qquad \forall x \in D,
\end{aligned}
$$

where $\gamma > 0$, $\omega > 0$, and $\alpha \in \mathbb{R}$ are the system parameters.

(i) Using the sufficient conditions for Lyapunov stability on a generic quadratic candidate function, find conditions on the system parameters for which the origin of the hybrid system is uniformly globally pre-asymptotically stable. Show your work in detail.

(ii) Confirm your answer to item (i) via simulations.

(iii) Is the origin uniformly globally asymptotically stable? Justify your answer.

2. Consider the bouncing ball example,

$$
f(x) = \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix}, \quad C = \left\{ x \in \mathbb{R}^2 \mid x_1 > 0 \right\}
$$

$$
g(x) = \begin{pmatrix} 0 \\ -e x_2 \end{pmatrix}, \quad D = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 < 0 \right\}
$$

where $e \in [0, 1)$. Use the Lyapunov function candidate

$$
V_2 = (1 + \theta \arctan x_2) \left(\frac{1}{2} x_2^2 + \gamma x_1 \right), \quad \theta = \frac{1 - e^2}{\pi (1 + e^2)}
$$

to

(i) show that the origin $(0, 0)$ is uniformly globally asymptotically stable.

(ii) in the plane, plot: a) the flow and jump sets, b) level sets of the Lyapunov functions, and c) a solution starting from $(1, 0)$ and a solution starting from $(0, -1)$. Show graphically that the solutions go from larger to smaller level sets of the Lyapunov function.

3. Consider the hybrid system with state $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, and

$$
f(x) := Ax \qquad \forall x \in C, \qquad\qquad g(x) := Ex \qquad \forall x \in D,
$$

where A, $E \in \mathbb{R}^{n \times n}$. Take the quadratic function

$$
V(x) = x^\top P x,
$$

where P is a positive definite matrix.

(i) What are the conditions on A and E so that $\mathcal{A} := \{x \in \mathbb{R}^n \mid x = 0\}$ (the origin) is globally uniformly pre-asymptotically stable (UGpAS)?

(ii) What additional conditions on the set of solutions to the hybrid system should be imposed so that $\mathcal{A} := \{x \in \mathbb{R}^n \mid x = 0\}$ is globally uniformly asymptotically stable (UGAS)?