MAE 286: Hybrid Systems (W14)
Homework #8

Due on 3/13/14

1. Consider the hybrid system on $\mathbb{R}^2$ with data

$$\begin{aligned} C &= \left\{x \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \geq 0\right\} \\ f(x) &= (1, 1.5) \\ D &= \left\{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\right\} \\ g(x) &= (-x_2/2, 0) \end{aligned}$$

Do the following

(i) Does the system satisfy the hybrid basic assumptions? Is it well-posed?

(ii) Use Lyapunov analysis to establish whether $\mathcal{A} = \{(0,0)\}$ is UGpAS.

(iii) Confirm your answer to (ii) via simulations.

2. Consider the hybrid system on $\mathbb{R}^2$ with data

$$\begin{aligned} C &= \left\{x \in \mathbb{R}^2 \mid x_2 > 0, x_1 \leq x_2^2\right\} \\ f(x) &= (1, 1.5) \\ D &= \left\{x \in \mathbb{R}^2 \mid x_2 \geq 0, x_1 = x_2^2\right\} \\ g(x) &= (-x_2/3, x_2^2) \end{aligned}$$

Do the following

(i) Does the system satisfy the hybrid basic assumptions? Is it well-posed?

(ii) Is $\mathcal{A} = \{(0,0)\}$ locally pre-asymptotically stable for $\mathcal{H}$? If so, what is its basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$?

(iii) Is $\mathcal{A} = \{(0,0)\}$ robustly $\mathcal{KL}$ pre-asymptotically stable on $\mathcal{B}^p_{\mathcal{A}}$ for $\mathcal{H}$? Justify your answer.

(iv) Does there exist a smooth Lyapunov function for $\mathcal{A}$ on $\mathcal{B}^p_{\mathcal{A}}$? Any idea how it looks like?

3. Read the brief discussion on switching systems in Chapter 1 ("Switching systems" within Section 1.4) and on generators of switching signals in Chapter 2 ("Generators for classes of switching signals", Section 2.4). Let $N_\sigma(s,t)$ denote the number of switching times of the signal $\sigma : [0,\infty) \longrightarrow Q$ in the time interval $[s,t]$. Consider the sets of signals defined by

$$\mathcal{S}_{\text{ave}}[\tau_D, N_0] = \left\{\sigma \mid \sigma \text{ piecewise constant signal such that } N_\sigma(s,t) \leq N_0 + \frac{t-s}{\tau_D}\right\},$$

$$\mathcal{S}_{\text{p-dwell}}[\tau_D, T] = \left\{\sigma \mid \sigma \text{ piecewise constant signal with infinitely many intervals of length} \geq \tau_D \right.$$
$$\left. \text{on which } \sigma \text{ is constant separated by no more than } T\right\}.$$

Do the following:

(i) Let $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$ and assume that there exist n consecutive discontinuities of σ separated by less than $\delta\tau_D$, with $\delta \in (0,1)$. Show that

$$n \leq \frac{N_0 - \delta}{1 - \delta}$$

(ii) Use (i) to deduce that

$$\mathcal{S}_{\text{ave}}[\tau_D, N_0] \subset \mathcal{S}_{\text{p-dwell}}\big[\delta\tau_D, \delta\tau_D \frac{N_0 - \delta}{1-\delta}\big].$$

4. (LaSalle Invariance Principle). Consider the following set of switching signals,

$$\mathcal{S}_{\text{weak-dwell}} = \bigcup_{\tau_D > 0} \mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty].$$

These are all the signals which have a positive persistent dwell-time. We refer to them as signals with weak dwell time. Consider the switched linear system

$$\dot{x} = A_\sigma x, \quad Q = \{1, 2\}, \quad \sigma \in \mathcal{S}_{\text{weak-dwell}},$$

where

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

Do the following:

(i) Plot the phase portrait of each system and show that $V(x) = x^T I x$ is a common Lyapunov function

(ii) Use the Invariance Principle to justify that the origin is an asymptotically stable equilibrium

(iii) Construct a switching signal $\sigma \notin \mathcal{S}_{\text{weak-dwell}}$ for which we do not have asymptotic stability

Hint. At time $t = 0$, consider a state in the axis $x_2 = 0$ and set $\sigma = 1$. The idea is to keep this signal for some time δ_k, and then switch to $\sigma = 2$ and keep it for $\delta_k/(1 - 2\delta_k)$, such that the state goes back to the axis $x_2 = 0$. Determine this state as a function of the original state. Then iterate this process and select δ_k such that an infinite time elapses without ever reaching the origin.

(iv) Construct a switching signal σ belonging to $\mathcal{S}_{\text{weak-dwell}}$ but not to $\mathcal{S}_{\text{p-dwell}}(\tau_D, T)$ for any $\tau_D > 0$ and $T < \infty$ for which we do not have uniform asymptotic stability.

Hint. Use the idea laid out in the previous hint to construct a switching signal with no finite period of persistence so that the convergence is arbitrarily slow.