Mathematical Analysis for Applications - MAE 289A - Final

Student name ___________________________________

Instructions

(i) Please read carefully the questions of the exam.

(ii) You have 24 hours to complete it since Thurs 13:00pm until Friday 13:00pm. Please return at EBU-I, 1807.

(iii) The exam is open notes.

(iv) Your solution must be done independently of others; discussing problem solutions with others is not allowed. The minimum suspicion this rule has been violated will invalidate your exam.

Questions

1. (15 points) Given a metric space (X, d), prove that the following are equivalent:
   (i) (X, d) is complete
   (ii) Any sequence $C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_n \supset \dots$ of non-empty sets in X such that $\lim_{n\to\infty} \text{diam}(C_n) = 0$ satisfies that $\cap_{n=1}^{\infty} C_n \neq \emptyset$.

2. (15 points) Consider the problem:

$$\begin{aligned} &\min x^2 + 5y^3 - y \\ &\text{s.t. } x^2 + y^2 \leq 1, \\ &\quad\;\; y \geq 0. \end{aligned}$$

   (i) Can this be considered to be a convex problem?
   (ii) Define the associated Lagrangian and compute the solutions via the KKT conditions. Do we have a zero duality gap for this problem?

3. (15 points) Let $(\mathbb{R}^n, d_2)$ the Euclidean metric space induced by the norm $\|\cdot\|_2$. Prove the following statements about the sets:

$$A + B = \{a + b \mid a \in A,\, b \in B\}, \quad [A, B] = \cup\{[a, b] \mid a \in A,\, b \in B\}$$

   (i) If A is closed and B is compact, then $A + B$ is closed but the conclusion is false when only A and B are assumed to be closed.
   (ii) If A and B are compact, then $[A, B]$ is also compact.

4. (15 points) Consider the following:
   (i) Let $f : A \longrightarrow \mathbb{R}$ a continuous function defined over a closed and unbounded set $A \subseteq \mathbb{R}^n$. Prove that if the limit $\lim_{\|x\|_2 \to +\infty} f(x)$ exists and is equal to a finite value L, then f is uniformly continuous.
   (ii) Is the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2$ uniformly continuous on $\mathbb{R}$? Why / why not?

5. (15 points) Suppose that f is a real-valued function in a open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \dots, D_n f$ are bounded on E. Prove that f is continuous on E.