MAE 289A: Mathematical Analysis for Applications (F15)

Homework #2

Due on 10/8/15

1. Let A be a nonempty set of real numbers which is bounded above. Let $-A$ be the set defined by

$$-A = \{-x \in \mathbb{R} \mid x \in A\}$$

   How are $\inf(-A)$ and $\sup(A)$ related?

   Hint: Draw some pictures in the real line, to have an idea, and then prove it. What about the relationship between $\sup(-S)$ and $\inf(S)$, where S is a set which is bounded below?

2. Let $b > 1$ and $y > 0$. In this exercise, we prove that there is a unique real number x (the *logarithm of* y *to the base* b) such that $b^x = y$ by completing the following outline:

   (i) For any positive integer n, $b^n - 1 \geq n(b-1)$.

   (ii) Hence $b - 1 \geq n(b^{1/n} - 1)$.

   (iii) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

   (iv) If w is such that $0 < b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply the previous part with $t = y \cdot b^{-w}$.

   (v) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

   (vi) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

   (vii) Prove that x is unique.

   Note: To do this problem, use that b^x, for $b > 1$ and $x \in \mathbb{R}$, is a well-defined operation that satisfies $b^{x+y} = b^x b^y$, for every $x, y \in \mathbb{R}$.

3. Prove that for any two complex numbers z, x,

$$||z| - |x|| \leq |z - x|.$$

4. Consider a paper square of $30 \times 30\text{cm}^2$ and cut out equal-sized small squares from each corner. After this, fold up the sides to form a box. What would be the maximum volume that the box can have as a function of the small square side? (You can not use derivatives in your solution!)

   Hint: Use the arithmetic-geometric mean inequality

5. Consider the set of all numbers of the form $a + b\sqrt{7}$, with a and b rational. Is this a field? Justify your answer.