MAE 289A: Mathematical Analysis for Applications (F15)

Homework #5

Due on 11/5/15

1. Let X be the space of $n \times n$ matrices, equipped with the induced L_2 norm (i.e., $\|A\|_2 = \sup_{x \neq 0} \|Ax\|_2 / \|x\|_2$, where $\|x\|_2$ is the Euclidean norm on $\mathbb{R}^n$). Let $S_n^{\geq}$ be the subset of X consisting of positive semidefinite, symmetric matrices. Prove that $S_n^{\geq}$ is a pointed, proper, convex cone.

   (A cone K is pointed if $K \cap -K = 0$, and solid if the interior of K is not empty. A cone that is convex, closed, pointed and solid is called a proper cone.)

2. Let $S \subset \mathbb{R}$, $S \neq \emptyset$ with $z = \inf(S) > -\infty$. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ such that $x_n \to z$.

3. Suppose $\{p_n\}_{n=1}^{\infty} \subset A$ does not have any convergent subsequences. Prove that for any $p_k \in \{p_n\}_{n=1}^{\infty}$, there exists $\delta > 0$ such that $B_\delta(p_k) \cap \{p_n\}_{n=1}^{\infty} = \{p_k\}$ (the set consisting only of the point p_k).

   Hint: Reason by contradiction.

4. Let $s_1 = \sqrt{2}$ and define

   $$s_{n+1} = \sqrt{2 + \sqrt{s_n}}, \qquad n \in \mathbb{N}.$$

   Prove that $s_n < 2$ for all $n \in \mathbb{N}$ and that $\{s_n\}$ converges.

   Hint: Use induction and prove that the sequence is monotonic.

5. Suppose that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to a point $p \in X$. Prove that the whole sequence $\{p_n\}_{n=1}^{\infty}$ converges to p.

6. Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be Cauchy sequences in a metric space X. Show that the sequence $\{d(q_n, p_n)\}_{n=1}^{\infty}$ converges.

   Hint: Note that, for any $m, n \in \mathbb{N}$, by the triangle inequality,

   $$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

   Therefore, it follows that $|d(p_n, q_n) - d(p_m, q_m)|$ is small if m and n are large. Use this fact!