MAE 289A: Mathematical Analysis for Applications (F15)

Homework #6

Due on 11/12/15

1. Recall the Netwon method described in class to find a numerical representation of $\sqrt{3}$ by solving the equation $g(x) = 0$, where $g(x) = x^2 - 3$. After some manipulations, we arrived at

$$x_n = x_{n-1} - \frac{x_{n-1}^2 - 3}{2x_{n-1}}. \tag{1}$$

As we reasoned, this can be posed as fixed-point iteration for the function $f : X \to X$, $f(x) = x - (x^2 - 3)/2x$. Do the following:

   (i) For the choice $X = [t, \infty)$, find conditions on t that ensure $f(X) \subset X$. Is X a complete metric space?

   (ii) Find conditions on t that ensure that f is a contraction.

   (iii) Apply Banach's fixed-point theorem to establish the convergence properties of the iteration (1)

   (iv) How many iterations would you need to approximate $\sqrt{3}$ with an error less than or equal to 10^{-6} starting from $x_0 = 7$?

   (v) User your preferred programming method (e.g., Matlab/Mathematica/C++/Python) to implement the iteration (1) starting from $x_0 = 7$ and $x_0 = 2$.

2. If $\sum a_n$ converges absolutely and $\{b_n\}$ is a sequence of positive numbers monotonically increasing and convergent, prove that $\sum a_n b_n$ converges.

3. For what values of p does the series $\sum_{n=1}^{\infty} \frac{\log n}{n^p}$ converge?

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that

   (i) $f(E)$ is dense in $f(X)$;

   (ii) If $g(p) = f(p)$ for all $p \in E$, then $g(p) = f(p)$ for all $p \in X$.

   (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

5. Disprove the claim that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ maps open sets onto open sets, i.e., that if C is open, then $f(C)$ is open.