MAE 289A: Mathematical Analysis for Applications (F15)

Homework #8

Due on 12/3/15

1. Let $C_0, \dots, C_n$ be real constants and assume

   $$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0.$$

   Prove that the equation $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.

   Hint: Use integration over the last expression.

2. Let f be defined on $\mathbb{R}$ and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all x, y. Prove that f is constant.

   Hint: Compute the differential of the function.

3. Suppose f takes real values, is three times differentiable, defined on $[-1, 1]$, such that

   $$f(-1) = 0,\ f(0) = 0,\ f(1) = 1,\ f'(0) = 0$$

   Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

   Hint: Use Taylor's theorem, with $\alpha = 0$, $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.

4. *Running average of a convex function*. Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is convex and differentiable. Show that its running average $F : \mathbb{R}_{>0} \to \mathbb{R}$, defined as

   $$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

   is convex.

5. Consider the following optimization problem

   $$\begin{aligned} \text{minimize} \quad & f_0(x, y), \\ \text{subject to} \quad & 2x + y \geq 1, \\ & x + 3y \geq 1 \\ & x \geq 0, \quad y \geq 0, \end{aligned}$$

   with variable $(x, y) \in \mathbb{R}^2$. Make a sketch of the feasible set. For the functions (i) $f_0(x, y) = x + y$, (ii) $f_0(x, y) = x$ give the optimal set and the optimal value.