MAE140 - Linear Circuits - Fall 11
Final, December 7

Instructions

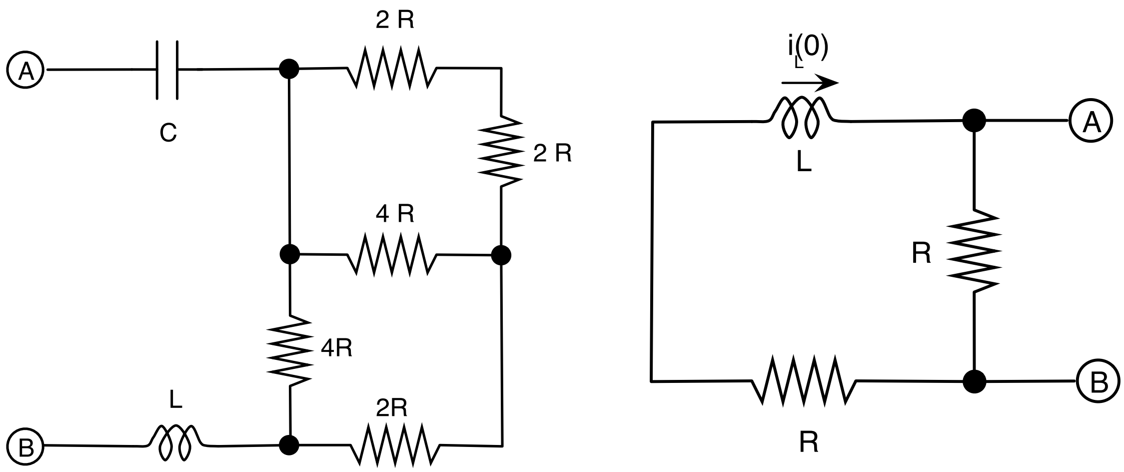
(i) The exam is open book. You may use your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 180 minutes

(iii) Do not forget to write your name, student number, and instructor

(iv) On the questions for which we have given the answers, please provide detailed derivations.

(v) The exam has 6 questions for a total of 60 points and 5 bonus points



(a) Question 1, Part I

(b) Question 1, Part II

Figure 1: Circuits for Question 1.

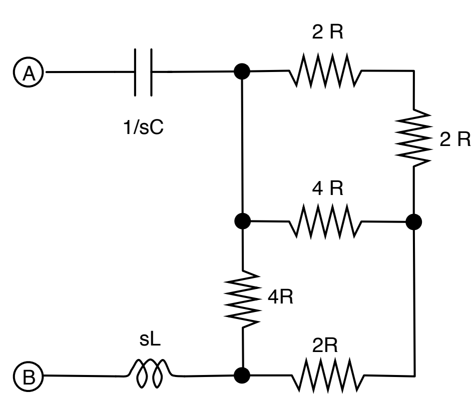
1. **Equivalent Circuits**

Part I: [5 points] Assuming zero initial conditions, transform the circuit into the s-domain and find the impedance equivalent to the circuit in Figure 1(a) as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Part II: [5 points] Assuming that the initial condition of the inductor is as indicated in the diagram, redraw the circuit shown in Figure 1(b) in the s-domain. Then use source transformation to find the s-domain Norton equivalent of this circuit as seen from terminals A and B.

Solution: Part I:

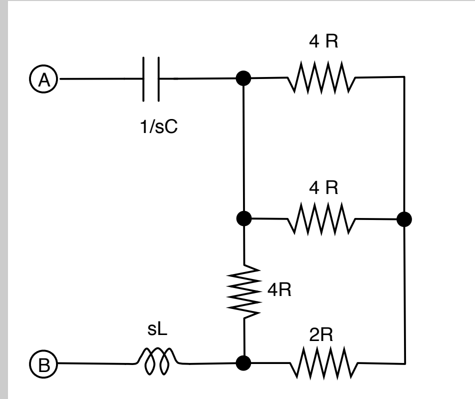
Since all initial conditions are zero, it is easy to transform the circuit to the s-domain.



[1 point]

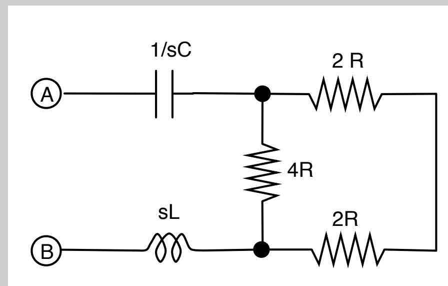
We now need to go through the different combinations of impedances carefully. We begin by combining the two resistors in series to get

[.5 point]



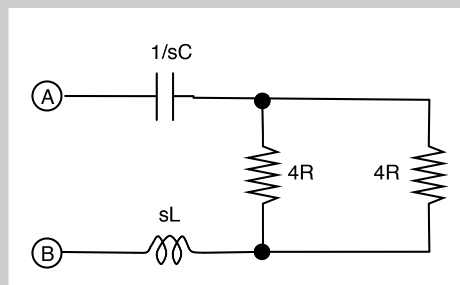
Now we combine the two resistors in parallel to get

[1 point]



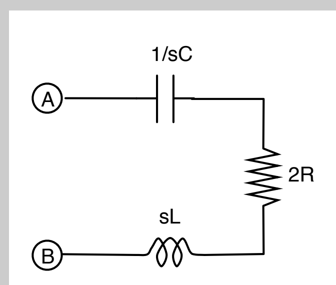
Now we combine the two resistors in series to get

[1 point]



Now we combine the two resistors in parallel to get

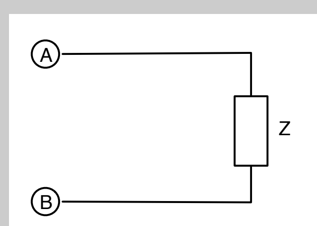
[.5 point]



Finally, we combine all the impedances in series to get

$$Z(s) = \frac{1}{sC} + sL + 2R = \frac{1 + s^2LC + 2RCs}{sC}$$

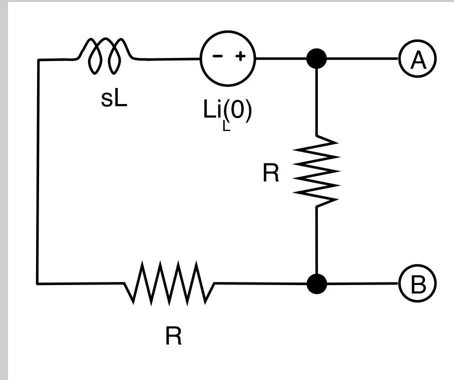
[1 point]



Part II:

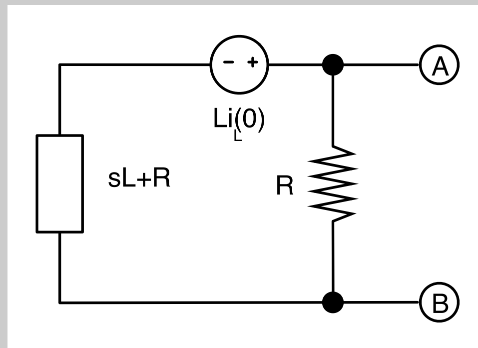
We begin by transforming the circuit into the s-domain. We use a voltage source for the initial condition of the inductor – this choice makes our life easier later.

[2 points]



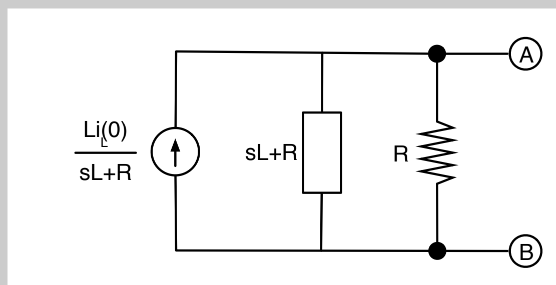
Now, we combine the inductor and the resistor in series

[1 point]



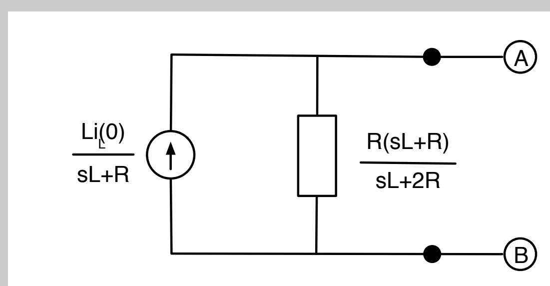
Next, we do a source transformation to turn the voltage source into a current source

[1 point]



Finally, we combine the impedance and resistor in parallel to get the Norton equivalent

[1 point]



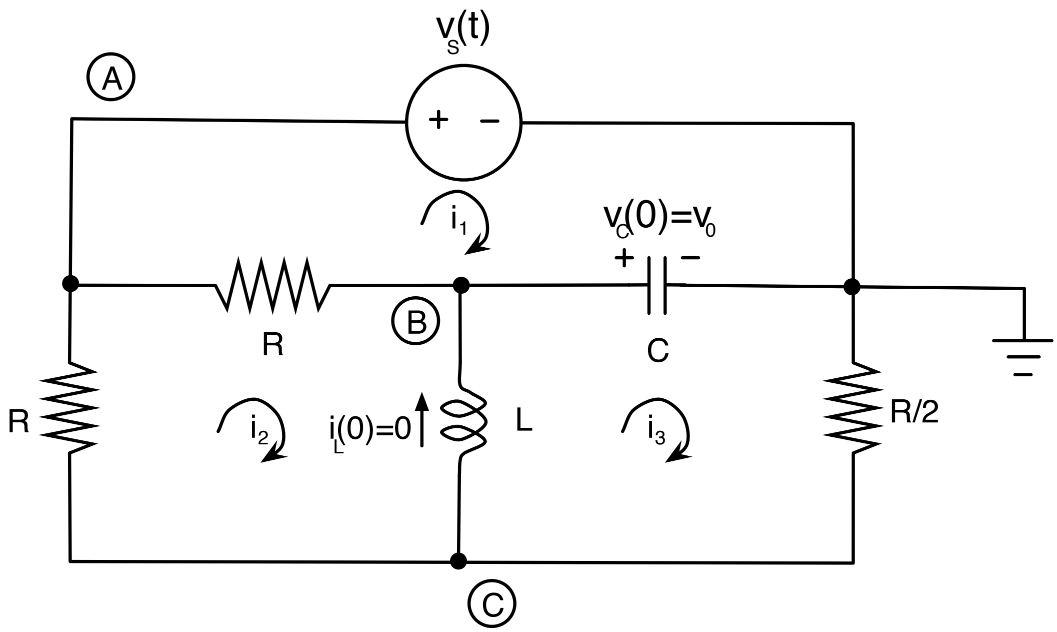


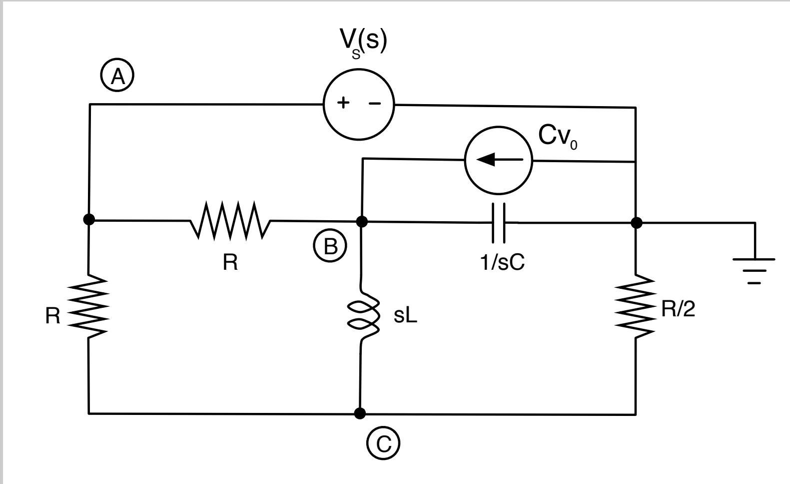
Figure 2: Nodal and Mesh Analysis Circuit

2. **Nodal and Mesh Analysis**

   Part I: [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into current sources. No need to solve any equations!

   Part II: [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the currents shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into voltage sources. No need to solve any equations!

Solution: Part I:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity and current orientation.

[1 point for correct circuit; 1 point for correct initial conditions]

The voltage source poses a problem for nodal analysis. We can easily take care of it by realizing that (method #2)

$$V_A(s) = V_S(s), \qquad \textbf{(1 point)}$$

and not writing KCL for node A.

Then, we only need to write KCL node equations for nodes B and C. For node B, we have

$$sC\,V_B(s) + \frac{1}{R}(V_B(s) - V_A(s)) + \frac{1}{sL}(V_B(s) - V_C(s)) = Cv_0 \qquad \textbf{(1 point)}$$

For node C, we have

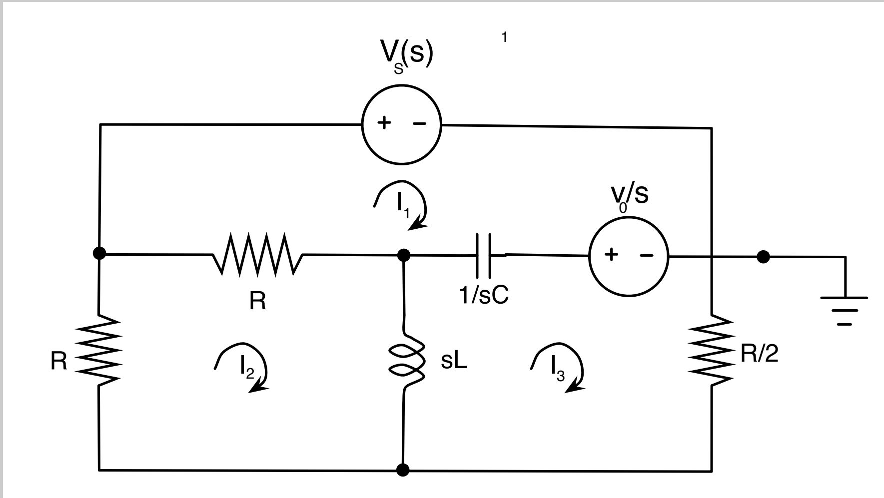
$$\frac{1}{R}(V_C(s) - V_A(s)) + \frac{1}{sL}(V_C(s) - V_B(s)) + \frac{2}{R}V_C(s) = 0 \qquad \textbf{(1 point)}$$

In matrix form, this looks like

$$\begin{pmatrix} sC + \frac{1}{R} + \frac{1}{sL} & -\frac{1}{sL} \\ -\frac{1}{sL} & \frac{1}{sL} + \frac{3}{R} \end{pmatrix} \begin{pmatrix} V_B(s) \\ V_C(s) \end{pmatrix} = \begin{pmatrix} Cv_0 + \frac{1}{R}V_S(s) \\ \frac{1}{R}V_S(s) \end{pmatrix}$$

which would have to solved in the unknowns $V_B(s)$ and $V_C(s)$.

Part II:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity and current orientation.

[1 point for correct circuit; 1 point for correct initial conditions]

We need to write mesh equations for meshes 1, 2, 3. For mesh 1, we have

$$R(I_1(s) - I_2(s)) + \frac{1}{sC}(I_1(s) - I_3(s)) = -V_S(s) + \frac{v_0}{s} \qquad \textbf{(1 point)}$$

For mesh 2, we have

$$R(I_2(s) - I_1(s)) + RI_2(s) + sL(I_2(s) - I_3(s)) = 0 \qquad \textbf{(1 point)}$$

For mesh 3, we have

$$sL(I_3(s) - I_2(s)) + \frac{R}{2}I_3(s) + \frac{1}{sC}(I_3(s) - I_1(s)) = -\frac{v_0}{s} \qquad \textbf{(1 point)}$$

In matrix form, this looks like

$$\begin{pmatrix} R + \frac{1}{sC} & -R & -\frac{1}{sC} \\ -R & 2R + sL & -sL \\ -\frac{1}{sC} & -sL & R/2 + sL + \frac{1}{sC} \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix} = \begin{pmatrix} -V_S(s) + \frac{v_0}{s} \\ 0 \\ -\frac{v_0}{s} \end{pmatrix}$$

which would have to solved in the unknowns $I_1(s)$, $I_2(s)$ and $I_3(s)$.

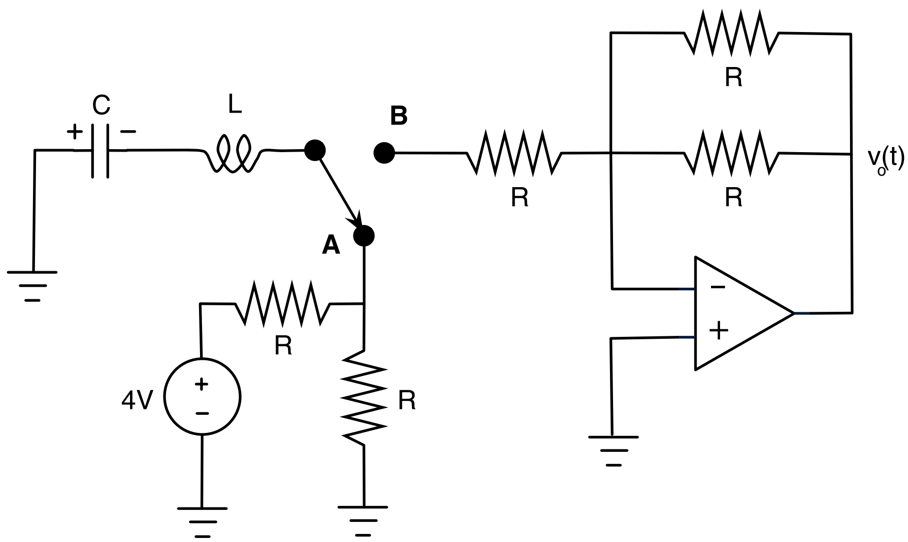


Figure 3: RCL circuit for Laplace Analysis

3. **Laplace Domain Circuit Analysis**

Part I: [3 points] Consider the circuit depicted in Figure 3. The voltage source is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial capacitor voltage and inductor currents are given by

$$v_C(0^-) = -2V, \quad i_L(0^-) = 0A.$$

[Show your work]

Part II: [2 points] Use these initial conditions to transform the circuit into the s-domain for $t \geq 0$. Use equivalent models for the capacitor and the inductor in which the initial conditions appear as voltage sources.
[Show your work]

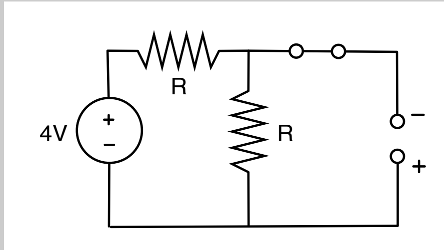
Part III: [5 points] Use domain circuit analysis and inverse Laplace transforms to show that the output voltage $v_o(t)$ when $C = \frac{1}{2}F$, $L = 6H$, and $R = 8\Omega$ is

$$v_o(t) = (2e^{-t} - 2e^{-t/3})u(t).$$

Solution:

Part I:

To find the initial conditions, we substitute the inductor by a short circuit and the capacitor by an open circuit.



[1 point]

Then it is clear from the circuit that

$$i_L(0^-) = 0 \qquad \textbf{(1 point)}$$

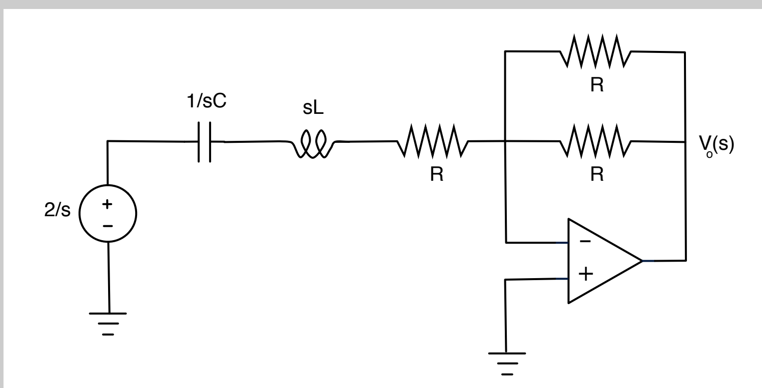
On the other hand, we find by voltage division that

$$v_C(0^-) = -\frac{R}{R+R}4 = -2V. \qquad \textbf{(1 point)}$$

Part II:

Since there is no initial current through the inductor, we do not need to add an independent voltage source for it. **[1 point]**

We add one voltage source in series for the capacitor to take care of its initial condition, paying special attention to the polarities.



[1 point]

Part III:

From our answer to Part II, we can see that the circuit in the s-domain corresponds to an inverting OpAmp and, therefore, the transform of the output voltage is

$$V_o(s) = -\frac{Z_2(s)}{Z_1(s)}\frac{2}{s} \qquad \textbf{(1 point)}$$

with impedances

$$Z_1(s) = \frac{1}{sC} + sL + R = \frac{LCs^2 + RCs + 1}{sC}$$
$$Z_2(s) = R \| R = \frac{R}{2} \qquad \textbf{(1 point)}$$

Substituting into $V_o(s)$, we get

$$V_o(s) = -\frac{RC}{LCs^2 + RCs + 1} = -\frac{4}{3s^2 + 4s + 1}$$

To find the output voltage, we need to compute the inverse Laplace transform. Using partial fractions, we set

$$V_o(s) = \frac{-4}{3s^2 + 4s + 1} = \frac{k_1}{s+1} + \frac{k_2}{s + \frac{1}{3}} \qquad \textbf{(.5 points)}$$

You can use your preferred method to find k_1 and k_2. We use here the cover-up or residue method

$$k_1 = \lim_{s \to -1} (s+1) V_o(s) = \lim_{s \to -1} \frac{-4}{3(s + \frac{1}{3})} = 2$$
$$k_2 = \lim_{s \to -\frac{1}{3}} (s + \frac{1}{3}) V_o(s) = \lim_{s \to -\frac{1}{3}} \frac{-4}{3(s+1)} = -2$$

Therefore, we have

$$V_o(s) = \frac{2}{s+1} - \frac{2}{s + \frac{1}{3}} \qquad \textbf{(2 points)}$$

The output voltage is then

$$v_o(t) = (2e^{-t} - 2e^{-t/3})u(t) \qquad \textbf{(.5 points)}$$

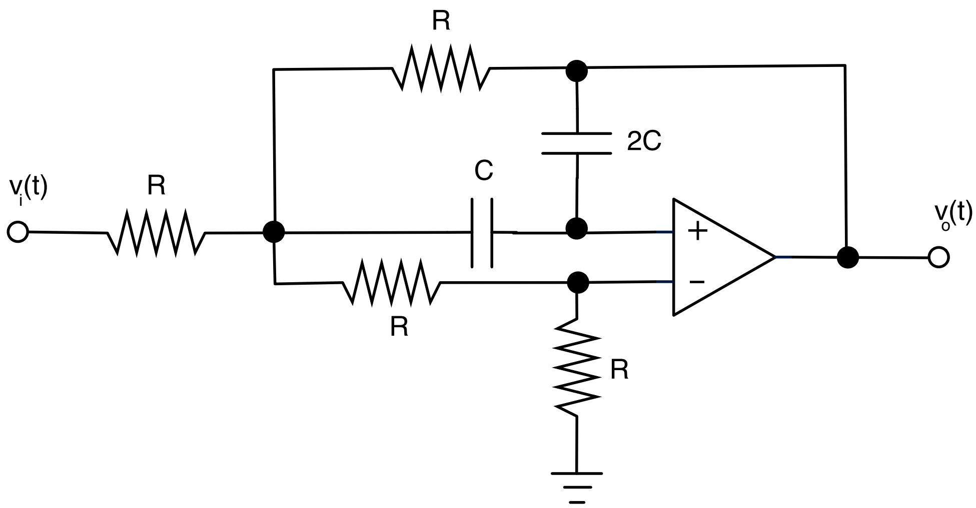
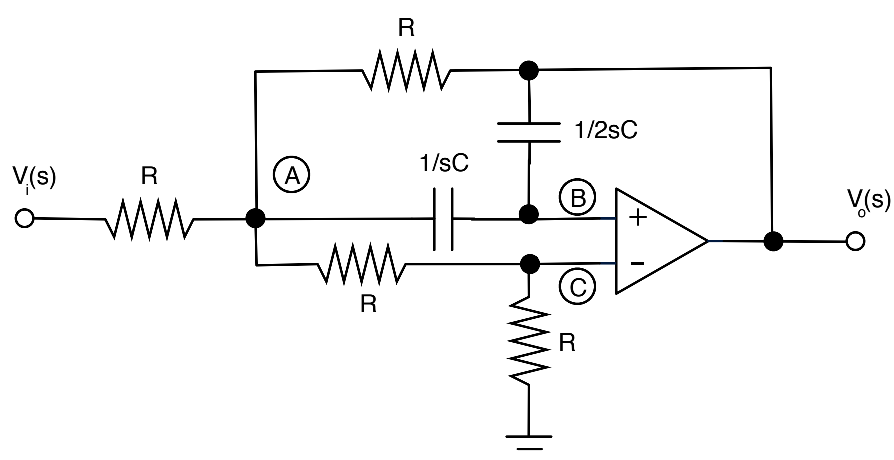


Figure 4: Frequency Response Analysis.

4. **Frequency Response Analysis**

Part I: [1 point] Assuming zero initial conditions, transform the circuit in Figure 4 into the s-domain.

Solution: Since all initial conditions are zero, there is no need to add an independent source for the capacitors. Therefore, the circuit in the s-domain looks like



[1 point]

Part II: [4 points] Show that the transfer function from $V_i(s)$ to $V_o(s)$ is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{9 + 2RCs}.$$

[Show your work]

Hint: use node voltage analysis

Solution: Since we cannot recognize any of the basic building blocks of OpAmps, we resort to nodal analysis. Nodal analysis at node A gives

$$\frac{1}{R}(V_A(s) - V_i(s)) + sC(V_A(s) - V_B(s)) + \frac{1}{R}(V_A(s) - V_C(s)) + \frac{1}{R}(V_A(s) - V_o(s)) = 0$$

(1 point)

Nodal analysis at node B gives

$$sC(V_B(s) - V_A(s)) + 2sC(V_B(s) - V_o(s)) = 0 \qquad \textbf{(1 point)}$$

Nodal analysis at node C gives

$$\frac{1}{R}(V_C(s) - V_A(s)) + \frac{1}{R}V_C(s) = 0 \qquad \textbf{(1 point)}$$

Additionally, the ideal OpAmp conditions give

$$V_B(s) = V_C(s) \qquad \textbf{(1 point)}$$

Substituting this equality into the other three, and solving for V_A, V_C, and V_o, we get

$$V_o(s) = \frac{1}{9 + 2RCs}V_i(s)$$

from which the answer follows.

Part III [3 points] Let $R = 10$ KΩ, $C = 100$ μF. Compute the gain and phase functions of $T(s)$. What are the DC gain and the ∞-freq gain? What is the cut-off frequency ω_c? Use these values to sketch the magnitude of the frequency response of the circuit. Is this circuit a low-pass, high-pass, or band-pass filter?
[Explain your answer]

Solution: If $R = 10$ KΩ, $C = 100$ μF, the transfer function takes the form

$$T(s) = \frac{1}{9 + 2s}$$

The frequency response is then the complex function

$$T(j\omega) = \frac{1}{9 + 2j\omega}, \quad \omega \geq 0$$

Its magnitude is the gain function,

$$|T(j\omega)| = \frac{1}{|9 + 2j\omega|} = \frac{1}{\sqrt{81 + 4\omega^2}} \qquad \textbf{(.5 point)}$$

And its phase is

$$\angle T(j\omega) = \angle 1 - \angle(9 + 2j\omega) = 0 - \arctan\left(\frac{2\omega}{9}\right) \qquad \textbf{(.5 point)}$$

At $\omega = 0$, we obtain

$$|T(j0)| = \frac{1}{9}, \quad \angle T(j0) = 0 \qquad \textbf{(.5 point)}$$

At $\omega = \infty$, we obtain

$$|T(j\infty)| = 0, \quad \angle T(j\infty) = -\frac{\pi}{2} \qquad \textbf{(.5 point)}$$

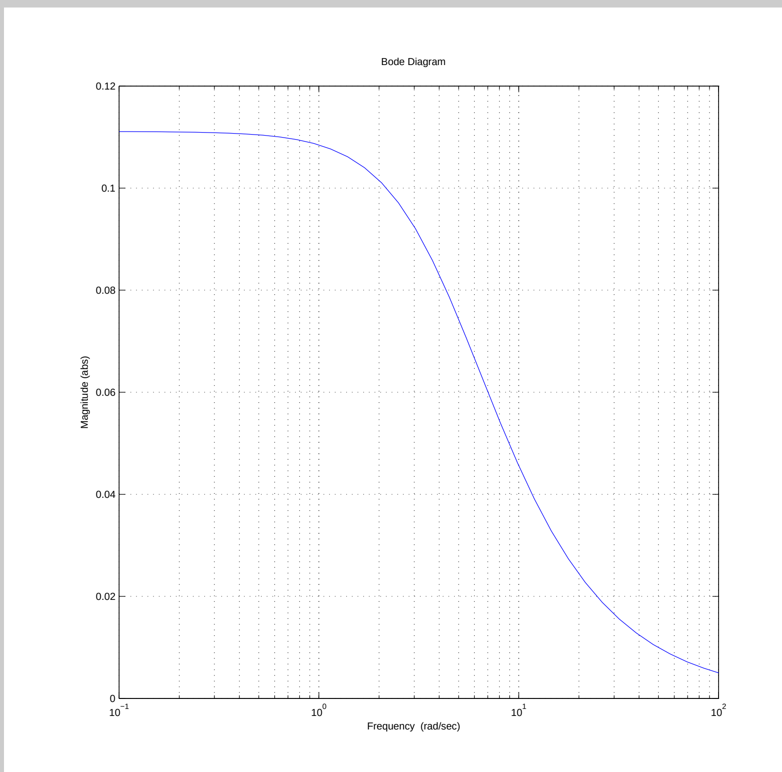
The cut-off frequency is defined by

$$|T(j\omega_c)| = \frac{T_{\text{max}}}{\sqrt{2}} = \frac{1}{9\sqrt{2}}.$$

Solving for it, we find $\omega_c = \frac{9}{2}$.

[.5 point]

With the values obtained above, you can sketch the magnitude of the frequency response as



This circuit is a low-pass filter.

[.5 point]

Part IV [2 points] Using what you know about frequency response, compute the steady state response $v_o^{SS}(t)$ of this circuit when $v_i(t) = 2\cos(\frac{9}{2}t + \frac{\pi}{2})$ using the same values of R and C as in Part III.

Solution: To compute the steady-state response to the input $v_i(t) = 2\cos(\frac{9}{2}t + \frac{\pi}{2})$, we use the frequency response values for $\omega = \frac{9}{2}$. In this way,

$$\begin{aligned} v_o^{SS}(t) &= 2\left|T\left(j\frac{9}{2}\right)\right|\cos\left(\frac{9}{2}t + \angle T(j\frac{9}{2}) + \frac{1}{2}\right) \\ &= 2\frac{1}{9\sqrt{2}}\cos\left(\frac{9}{2}t - \frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{\sqrt{2}}{9}\cos\left(\frac{9}{2}t + \frac{\pi}{4}\right) \end{aligned}$$

[1 point for correct expression]
[1 point for correct values]

5. **Active Filter Design**

Consider the transfer function

$$T(s) = \frac{V_0(s)}{V_i(s)} = \frac{2\lambda^2}{s^2 + 3\lambda s + 2\lambda^2}$$

where the parameter $\lambda \geq 0$ is to be specified by the user. In this question, always assume zero initial conditions.

Part I: [3 points] Show that the transfer function $T(s)$ can be realized as a product of two first-order low-pass filters of the form

$$T_1(s) = \frac{\pm\omega_1}{s + \omega_1}, \qquad T_2(s) = \frac{\pm\omega_2}{s + \omega_2}$$

that is, $T(s) = T_1(s) \times T_2(s)$. What is the cut-off frequency and gain of $T_1(s)$ and $T_2(s)$ in terms of λ?

Solution: First, factor the denominator of $T(s)$ as

$$s^2 + 3\lambda s + 2\lambda^2 = (s + \lambda)(s + 2\lambda)$$

Then,

$$T(s) = T_1(s)T_2(s), \qquad T_1(s) = \frac{\pm\lambda}{s + \lambda}, \qquad T_2(s) = \frac{\pm 2\lambda}{s + 2\lambda} \qquad \textbf{(1 point)}$$

The cut-off frequencies of each filter are

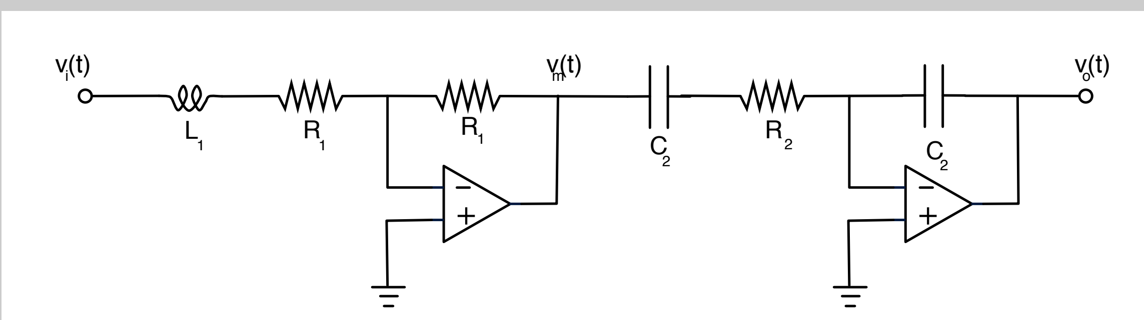
$$\omega_1 = \lambda, \qquad \omega_2 = 2\lambda. \qquad \textbf{(1 point)}$$

The gains are calculated at $s = 0j$ as

$$|T_1(0j)| = 1, \qquad |T_2(0j)| = 1. \qquad \textbf{(1 point)}$$

Part II: [4 points] Design a circuit that implements $T(s)$ as the product of the two filters $T_1(s)$ and $T_2(s)$ using no more than 2 OpAmps.

Solution: Many solutions are possible. Here is one that uses 2 inverting OpAmps,



The transfer function from $V_i(s)$ to $V_m(s)$ is

$$T_1(s) = -\frac{R_1}{R_1 + sL_1} = -\frac{\frac{R_1}{L_1}}{\frac{R_1}{L_1} + s}$$

The transfer function from $V_m(s)$ to $V_o(s)$ is

$$T_2(s) = -\frac{\frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} = -\frac{\frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}}$$

[4 points]

Part III: [3 points] Find values of the components in your design so that $\lambda = 2000\pi$ rad/s.

Solution: For the circuit designed in the previous part, we have

$$\omega_1 = \frac{R_1}{L_1} = \lambda, \qquad \omega_2 = \frac{1}{R_2C_2} = 2\lambda.$$

[1 point]

The set of components R_1, R_2, L_1, C_2 is then any combination that satisfies

$$\frac{R_1}{L_1} = 2000\pi, \quad R_2C_2 = \frac{1}{4000\pi}$$

[1 point]

For instance, for

$$L_1 = \frac{1}{\pi} \approx 318 \times 10^{-3} = 318 \text{ mH}$$
$$C_2 = \frac{1}{\pi} \times 10^{-9} \approx 318 \times 10^{-12} = 318 \text{ pF}$$

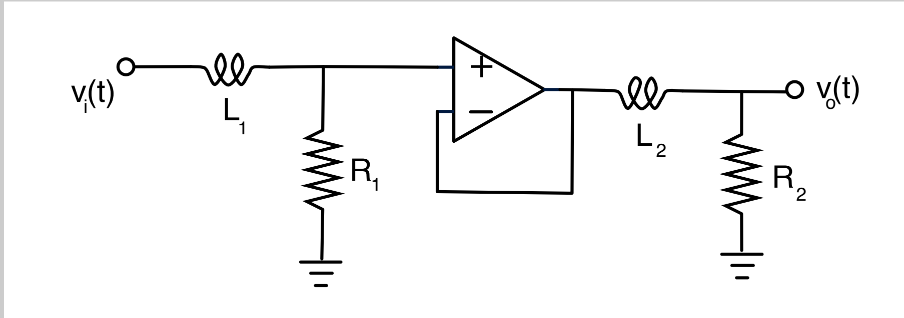
then

$$R_1 = 2000\pi L_1 = 2 \text{ K}\Omega$$
$$R_2 = \frac{1}{4000\pi C_2} = \frac{1}{4000 \times 10^{-9}} = .25 \times 10^6 = 250 \text{ K}\Omega$$

[1 point]

Part IV: [3 bonus points] Design a circuit that implements $T(s)$ as the product of the two filters $T_1(s)$ and $T_2(s)$ using only 1 OpAmp.

Solution: As before, there are multiple correct answers. A possible design is the following. A voltage divider with one resistor and one inductor conveniently placed has a transfer function of the form $T_1(s)$ (or $T_2(s)$ for that matter). So does a voltage divider with one capacitor and one resistor. We can use an OpAmp in a voltage follower configuration to connect two such circuits so that one does not load the other, as in the following figure



[3 bonus points]

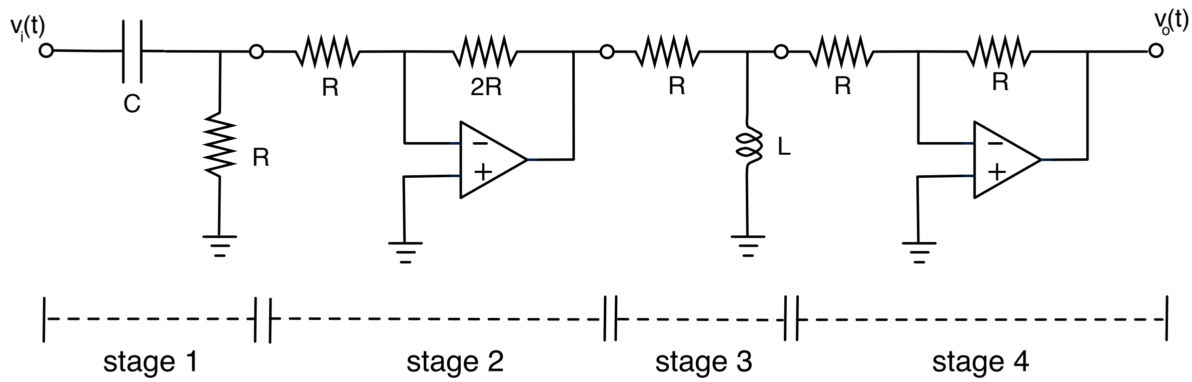


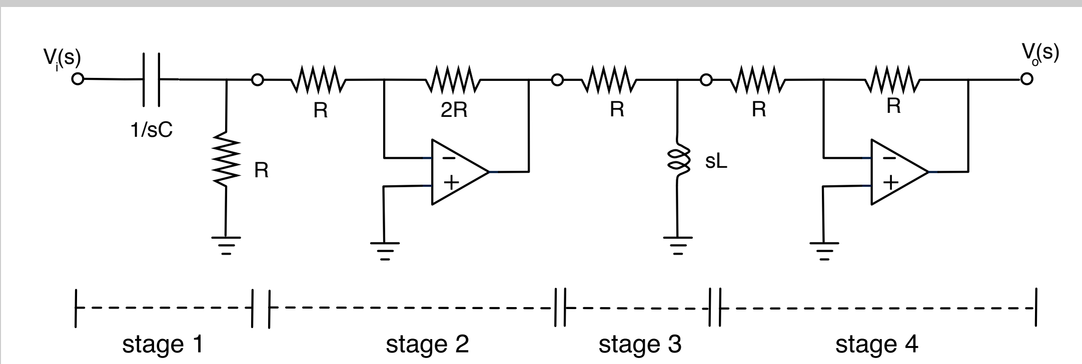
Figure 5: Circuit for Question 6.

6. **Chain rule and circuit design**

   Consider the circuit in Figure 5. You can assume zero initial conditions.

   Part I: [3 points] Redraw the circuit in Figure 5 in the s-domain and compute the transfer functions $T_1(s)$, $T_2(s)$, $T_3(s)$, $T_4(s)$ of each one of the stages.

Solution: With no initial conditions, the circuit in the s-domain simply looks like



[1 point]

Stage 1 is a voltage divider, hence

$$T_1(s) = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{RCs + 1} \qquad \textbf{(.5 point)}$$

Stage 2 is an inverting OpAmp, hence

$$T_2(s) = -\frac{2R}{R} = -2 \qquad \textbf{(.5 point)}$$

Stage 3 is a voltage divider, hence

$$T_3(s) = \frac{sL}{R + sL} \qquad \textbf{(.5 point)}$$

Stage 4 is an inverting OpAmp, hence

$$T_4(s) = -\frac{R}{R} = -1 \qquad \textbf{(.5 point)}$$

Part II: [2 points] Somebody with a rusty recollection of linear circuits analyzed the circuit in Figure 5 and concluded that the transfer function $T(s)$ from $V_i(s)$ to $V_o(s)$ is equal to the product of the transfer functions

$$\widetilde{T}(s) = T_1(s) \times T_2(s) \times T_3(s) \times T_4(s) = \frac{2RCLs^2}{RCLs^2 + (R^2C + L)s + R}$$

of the 4 stages identified in the plot. Identify two problems that invalidate this conclusion.

Solution: The two problems with the conclusion is that

(i) stage 2 is loading stage 1, **[1 point]**

(ii) stage 4 is loading stage 3, **[1 point]**

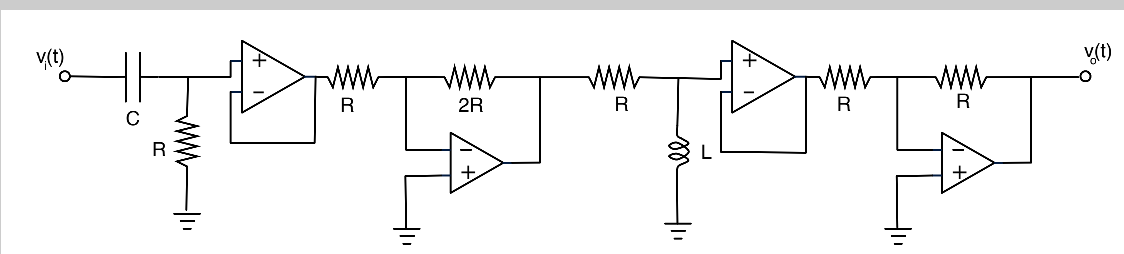
and hence the chain rule does not apply. This is because, in each case, there is current flowing through the input resistors of the inverting OpAmps.

Part III: [2 points] Modify Figure 5, keeping all 4 stages, so that the resulting circuit does have transfer function $\widetilde{T}(s)$ by adding at most 2 OpAmps.
[Justify your answer]

Solution: Given the problems identified in Part II, the easiest way to do this is by adding 1 OpAmp in a voltage follower configuration between stages 1 and 2, and another one between stages 3 and 4. The addition of these OpAmps makes the chain rule valid (since no stage $i+1$ would load stage i then).

[1 point]

This yields the design



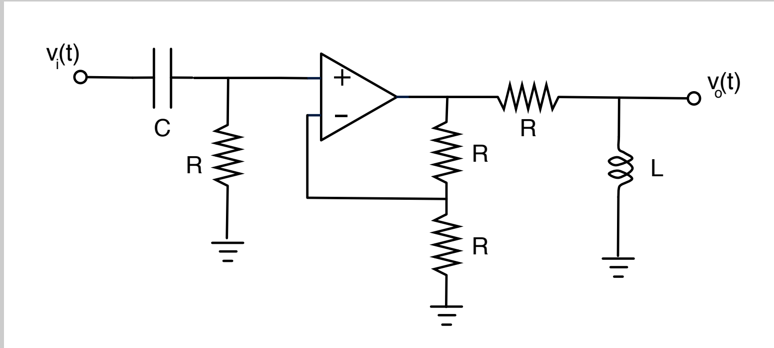
[1 point]

Part IV: [3 points] Use stage 1, stage 3 and a noninverting OpAmp to design yet another circuit with transfer function $\widetilde{T}(s)$.
[Provide reasons that justify how you arrived at your design]

Solution: As seen from Part I, the net effect of the two inverting OpAmps in Figure 5 is a gain of 2 in the transfer function $\widetilde{T}(s)$. We can also obtain this gain with a noninverting OpAmp.
[1 point]
Additionally, if we put this OpAmp connecting stages 1 and 3, then there is no load from stage 2 onto stage 1, or from stage 3 onto stage 2, and hence the chain rule applies.

[1 point]

Such a design would yield



[1 point]

Part V: [2 bonus points] What design would you recommend to realize the transfer function $\widetilde{T}(s)$, the answer in Part III or the answer in Part IV? Why?

Hint: you can provide at least 2 different reasons

Solution: The one in Part IV is better because it has a smaller number of components.

[1 bonus point]

The one in Part IV is also better because there is less potential for hitting the nonlinear OpAmp operation mode (since there is only 1 OpAmp, instead of 4), which would yield a different transfer function.

[1 bonus point]