

# Frequency Response

We now know how to analyze and design ccts via s-domain methods which yield dynamical information

- Zero-state response
- Zero-input response
- Natural response
- Forced response

The responses are described by the exponential modes

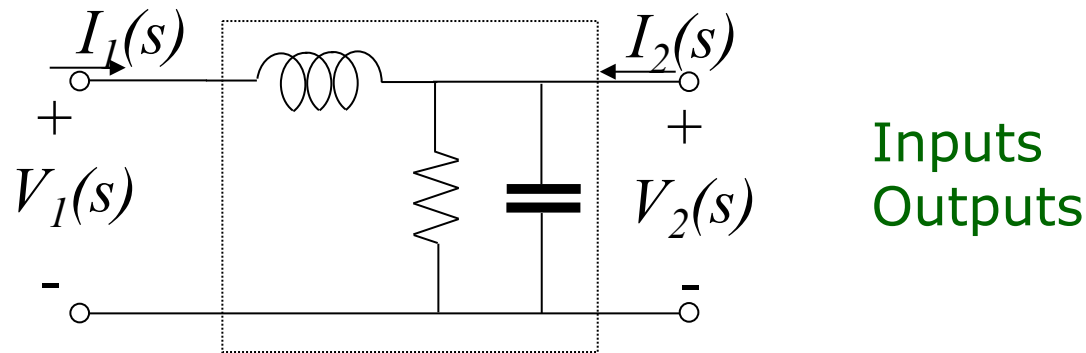
The modes are determined by the poles of the response Laplace Transform

We next will look at describing cct performance via frequency response methods

This guides us in specifying the cct pole and zero positions

# Transfer functions

Transfer function; measure input at one port, output at another



$$\text{Transfer function} = \frac{\text{zero - state response transform}}{\text{input signal transform}}$$

*(I.e., what the circuit does to your input)*

# Sinusoidal Steady-State Response

Consider a stable transfer function with a sinusoidal input  $x(t) = A \cos(\omega t + \phi)$

$$X(s) = A \frac{s \cos(\phi) - \omega \sin(\phi)}{s^2 + \omega^2}$$

The Laplace Transform of the response has poles

- Where the natural cct modes lie
  - These are in the open left half plane  $\text{Re}(s) < 0$
- At the input modes  $s = +j\omega$  and  $s = -j\omega$

Only the response due to the poles on the imaginary axis remains after a sufficiently long time

This is the sinusoidal steady-state response

# Sinusoidal Steady-State Response contd

**Input**  $x(t) = A \cos(\omega t + \phi) = A \cos \omega t \cos \phi - A \sin \omega t \sin \phi$

**Transform**  $X(s) = A \cos \phi \frac{s}{s^2 + \omega^2} - A \sin \phi \frac{\omega}{s^2 + \omega^2}$

## Response Transform

$$Y(s) = T(s)X(s) = \frac{k}{s - j\omega} + \frac{k^*}{s + j\omega} + \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \dots + \frac{k_N}{s - p_N}$$

## Response Signal

$$y(t) = \underbrace{ke^{j\omega t} + k^*e^{-j\omega t}}_{\text{forced response}} + \underbrace{k_1e^{p_1t} + k_2e^{p_2t} + \dots + k_Ne^{p_Nt}}_{\text{natural response}}$$

## Sinusoidal Steady State (SSS) Response

$$y_{SS}(t) = ke^{j\omega t} + k^*e^{-j\omega t}$$

# Sinusoidal Steady-State Response contd

Calculating the SSS response to  $x(t) = A \cos(\omega t + \phi)$

Residue calculation

$$\begin{aligned} k &= \lim_{s \rightarrow j\omega} [(s - j\omega)Y(s)] = \lim_{s \rightarrow j\omega} [(s - j\omega)T(s)X(s)] \\ &= \lim_{s \rightarrow j\omega} \left[ T(s)(s - j\omega)A \frac{s \cos \phi - \omega \sin \phi}{(s - j\omega)(s + j\omega)} \right] = T(j\omega)A \left[ \frac{j\omega \cos \phi - \omega \sin \phi}{2j\omega} \right] \\ &= \frac{1}{2} A e^{j\phi} T(j\omega) = \frac{1}{2} A |T(j\omega)| e^{j(\phi + \angle T(j\omega))} \end{aligned}$$

Signal calculation

$$\begin{aligned} y_{ss}(t) &= k e^{j\omega t} + k^* e^{-j\omega t} \\ &= |k| e^{j\angle k} e^{j\omega t} + |k| e^{-j\angle k} e^{-j\omega t} = 2|k| \cos(\omega t + \angle k) \end{aligned}$$

$$y_{ss}(t) = A |T(j\omega)| \cos(\omega t + \phi + \angle T(j\omega))$$

# Sinusoidal Steady-State Response contd

Response to  $x(t) = A \cos(\omega t + \phi)$

is  $y_{ss}(t) = A |T(j\omega)| \cos(\omega t + \phi + \angle T(j\omega))$

Output frequency = input frequency

Output amplitude = input amplitude  $\times |T(j\omega)|$

Output phase = input phase  $+ \angle T(j\omega)$

The Frequency Response of the transfer function  $T(s)$  is given by its evaluation as a function of a complex variable at  $s=j\omega$

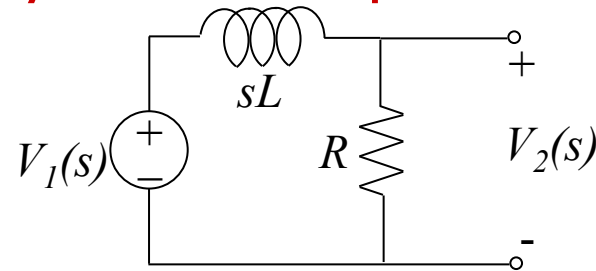
We speak of the amplitude response and of the phase response. They cannot independently be varied

$|T(j\omega)|$  *gain*

$\angle T(j\omega)$  *phase*

## Example 11-13, T&R 5th ed, p 527

Find the steady state output for  $v_1(t) = A \cos(\omega t + \phi)$



Compute the s-domain transfer function  $T(s)$

Voltage divider 
$$T(s) = \frac{R}{sL + R}$$

Compute the frequency response

$$|T(j\omega)| = \frac{R}{\sqrt{R^2 + (\omega L)^2}}, \quad \angle T(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Compute the steady state output

$$v_{2SS}(t) = \frac{AR}{\sqrt{R^2 + (\omega L)^2}} \cos\left[\omega t + \phi - \tan^{-1}(\omega L / R)\right]$$

# Terminology for Frequency Response

Based on shape of **gain function** of frequency

**Passband:** range of frequencies with nearly constant gain

**Stopband:** range of frequency with significantly reduced gain

**Cutoff frequency:** frequency associated with transition between bands

$$|T(j\omega_c)| = \frac{1}{\sqrt{2}} T_{\max}$$

**Low-pass filter:** passband plus stopband

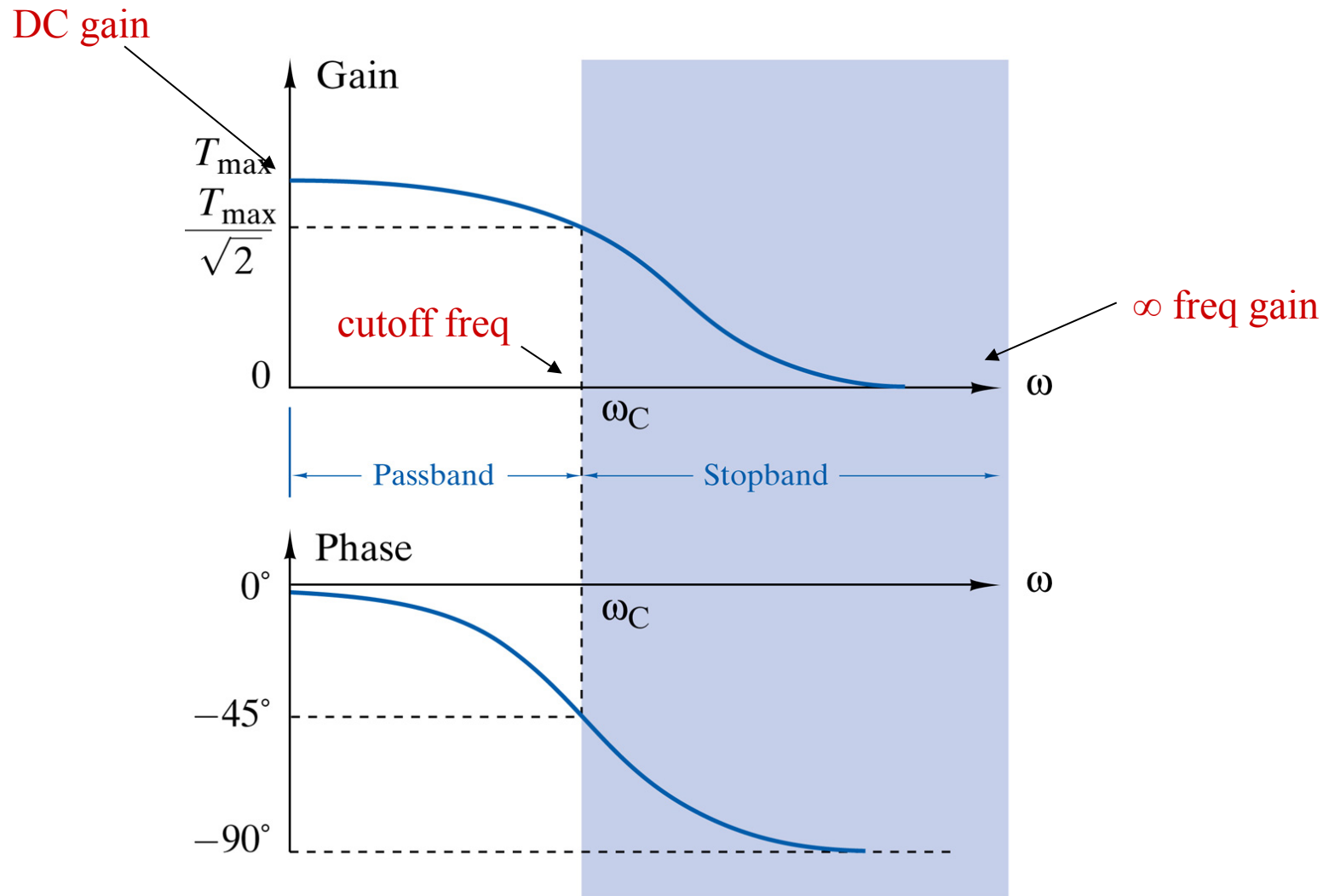
**High-pass filter:** stopband plus passband

**Bandpass filter:** one passband with two adjacent stopbands

**Bandstop filter:** one stopband with two adjacent passbands



# Terminology for Frequency Response



What kind of filter is this one?

# Worked-out example

$$T(s) = \frac{K}{s + \alpha}$$

What is DC gain? What is  $\infty$ -freq gain? What is cutoff freq?

$K, \alpha$  real,  $\alpha > 0$

First compute gain and phase

$$|T(j\omega)| = \frac{|K|}{\sqrt{\omega^2 + \alpha^2}}$$

$$\angle T(j\omega) = \angle K - \arctan\left(\frac{\omega}{\alpha}\right)$$

DC gain

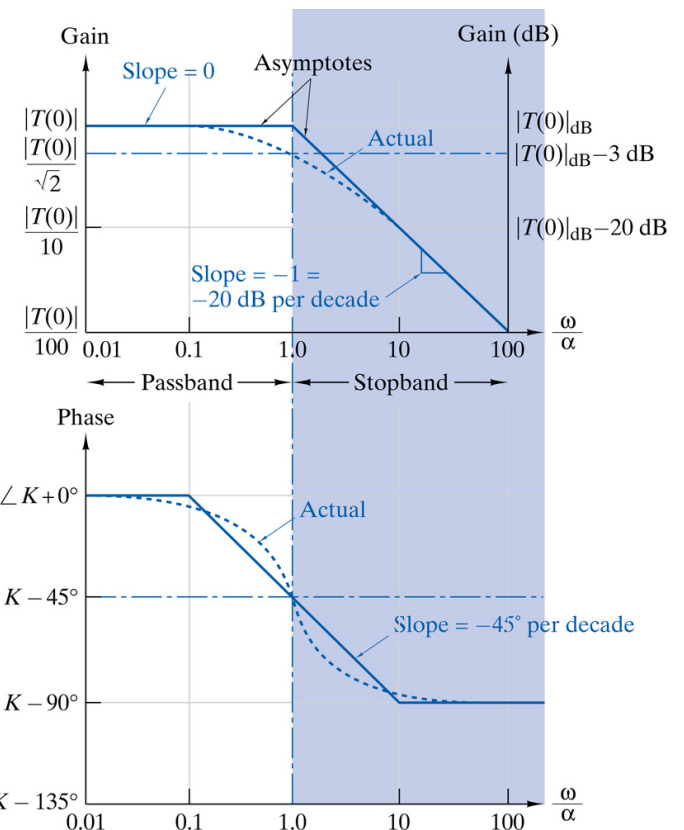
$$\lim_{\omega \rightarrow 0} |T(j\omega)| = \frac{|K|}{\alpha}$$

$\infty$ -freq

$$\lim_{\omega \rightarrow \infty} |T(j\omega)| = 0$$

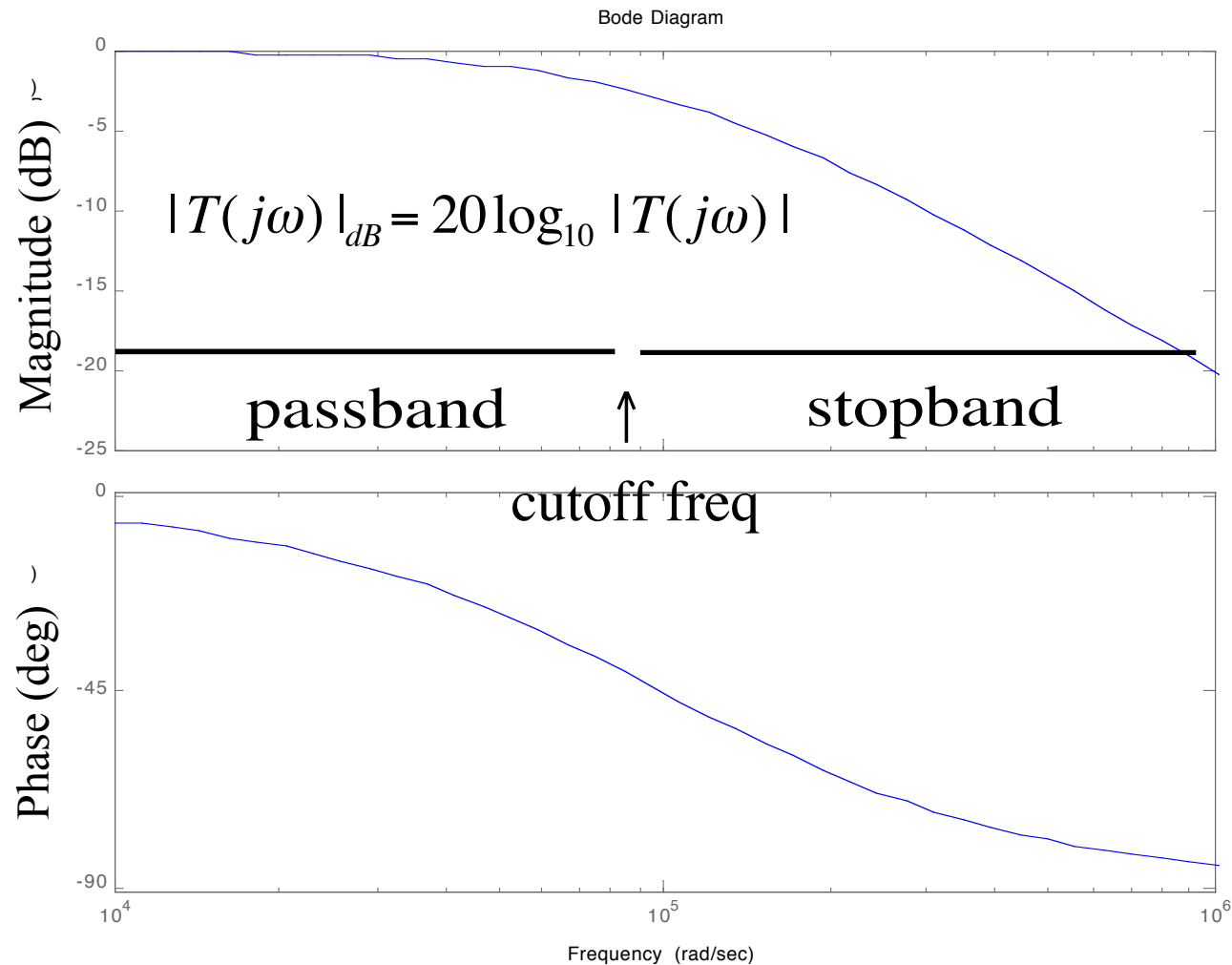
Cutoff freq

$$|T(j\omega_c)| = \frac{1}{\sqrt{2}} T_{\max} = \frac{1}{\sqrt{2}} \frac{|K|}{\alpha} \Rightarrow \omega_c = \alpha$$



# Frequency Response – Bode Diagrams

Log-log plot of  $\text{mag}(T)$ , log-linear plot  $\text{arg}(T)$  versus  $\omega$



# Matlab Commands for Bode Diagram

Specify component values

```
>> R=1000;L=0.01;
```

Set up transfer function

```
>> Z=tf(R,[L R])
```

Transfer function:

$$1000$$

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$$0.01 s + 1000$$

```
>> bode(Z)
```

# Frequency Response Descriptors

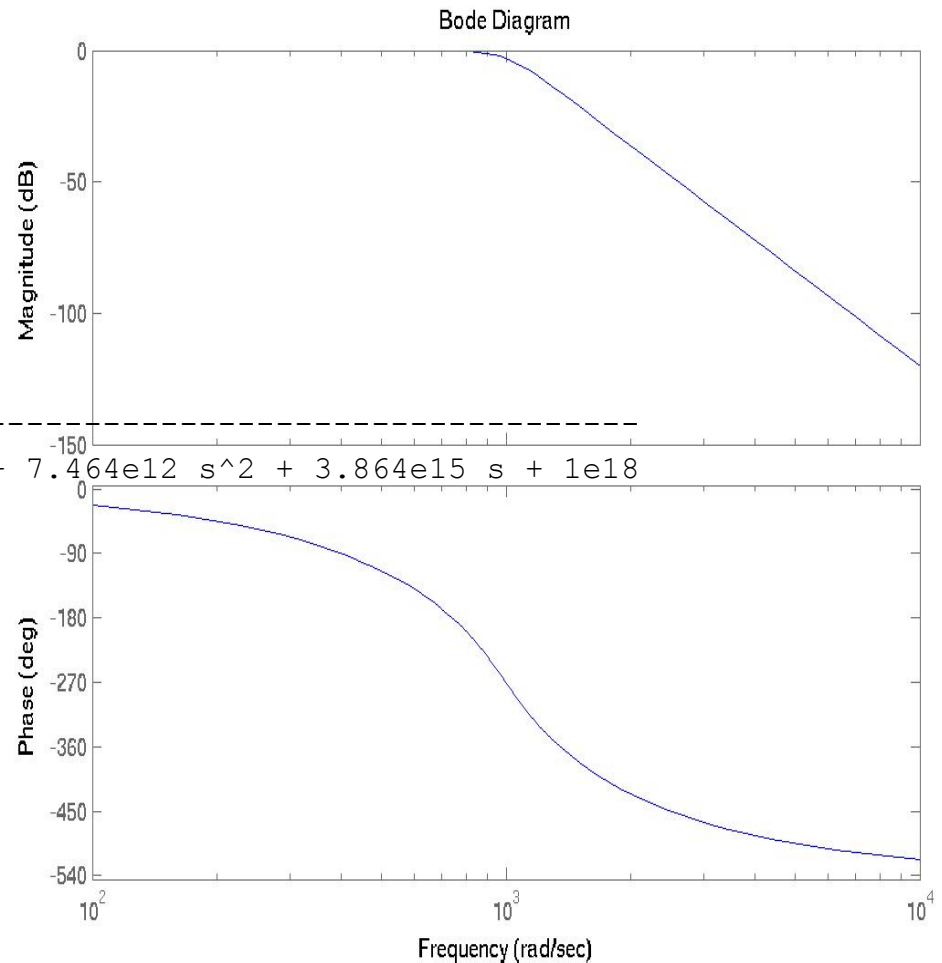
## Lowpass Filters

```
[num,den]=butter(6,1000,'s');  
lpass=tf(num,den);  
lpass
```

Transfer function:

$$\frac{1e18}{s^6 + 3864 s^5 + 7.464e06 s^4 + 9.142e09 s^3 + 7.464e12 s^2 + 3.864e15 s + 1e18}$$

```
bode(lpass)
```



# High Pass Filters

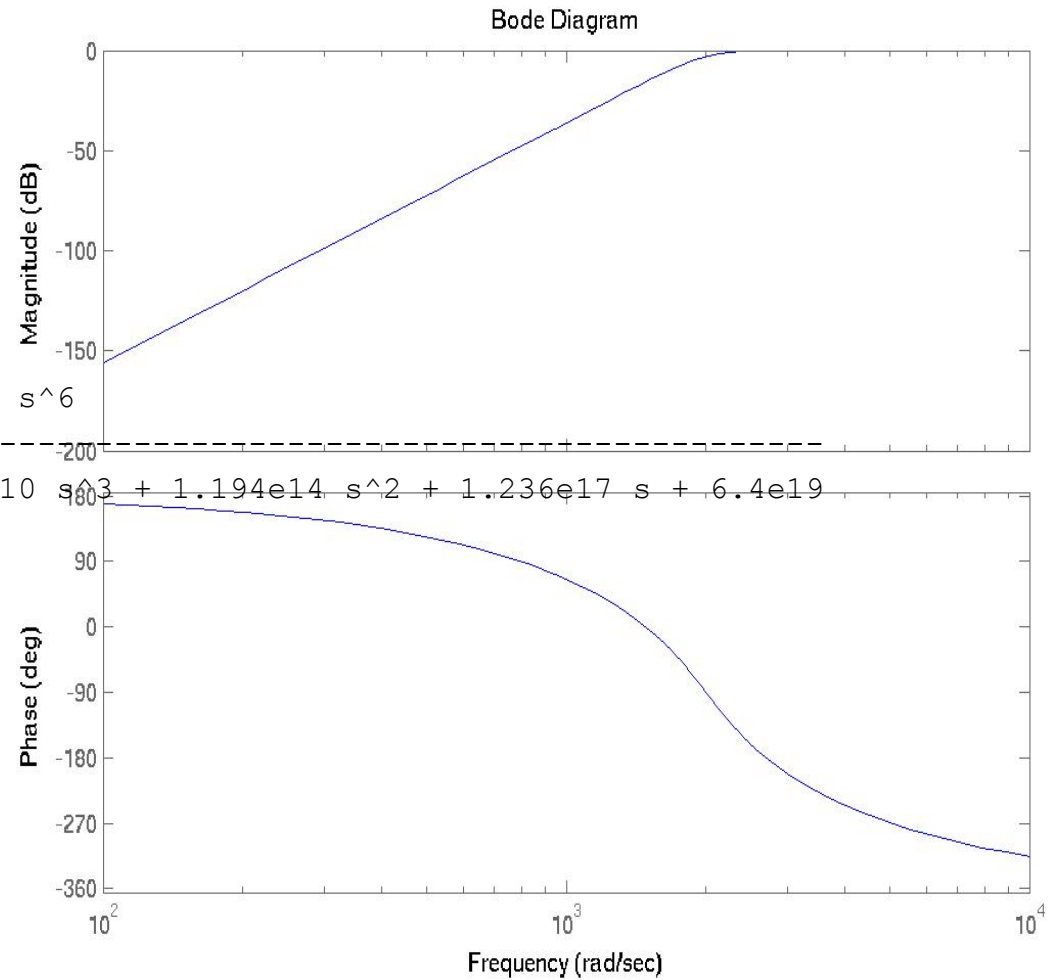
```
[num,den]=butter(6,2000,'high','s');  
hpass=tf(num,den)
```

Transfer function:

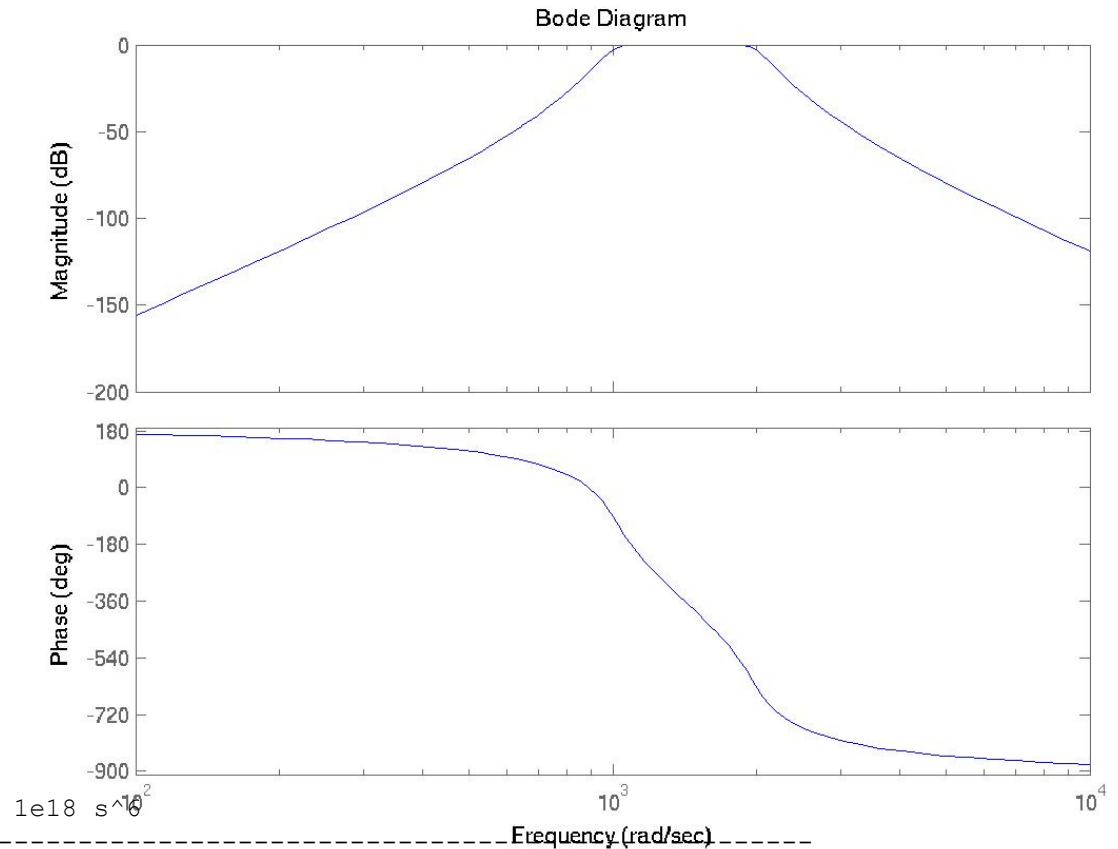
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$$s^6 + 7727 s^5 + 2.986e07 s^4 + 7.313e10 s^3 + 1.194e14 s^2 + 1.236e17 s + 6.4e19$$

```
bode(hpass)
```



# Bandpass Filters



```
[num,den]=butter(6,[1000 2000],'s');  
bpass=tf(num,den)
```

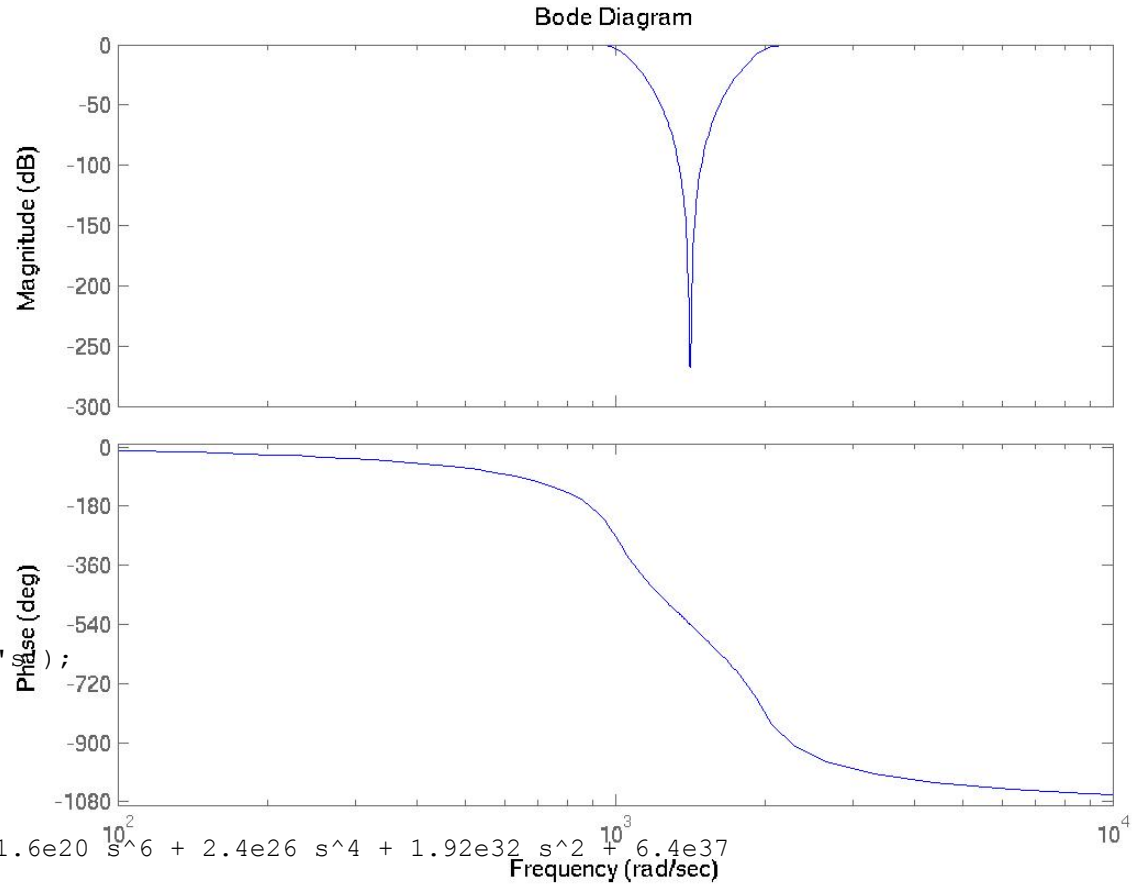
Transfer function:

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$$s^{12} + 3864 s^{11} + 1.946e07 s^{10} + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6$$
$$+ 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37$$

```
bode(bpass)
```

# Bandstop Filters



```
[num,den]=butter(6,[1000 2000],'stop','s');
bstop=tf(num,den)
```

Transfer function:

$$\frac{s^{12} + 1.2e07 s^{10} + 6e13 s^8 + 1.6e20 s^6 + 2.4e26 s^4 + 1.92e32 s^2 + 6.4e37}{s^{12} + 3864 s^{11} + 1.946e07 s^{10} + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6 + 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37}$$

```
bode(bstop)
```