MAE 140 HW6 Solution

9-1 Laplace of $f(t) = 500(1 - e^{-100t})u(t)$, find zeros and poles!

$$F(s) = \mathcal{L}\{500(1-e^{-100t})u(t)\} = 500\left(\mathcal{L}\{u(t)\} - \mathcal{L}\{e^{-100t}u(t)\}\right)$$

$$= 500\left(\frac{1}{s} - \frac{1}{s+100}\right) = \boxed{500\left(\frac{100}{s(s+100)}\right)}$$

Using the table in the book.

$$\boxed{\text{zeros} = \{\infty, \infty\}, \ \text{poles} = \{0, -100\}}$$

9-4 Laplace of $f(t) = 20(e^{-200t} - 2e^{-100t})$, find zeros and poles!

$$F(s) = \mathcal{L}\{20(e^{-200t} - 2e^{-100t})u(t)\} = 20\left(\frac{1}{s+200} - 2\frac{1}{s+100}\right)$$

$$= \boxed{20\left(\frac{-(s+300)}{(s+100)(s+200)}\right)} \rightarrow \boxed{\text{zeros} = \{-300, \infty\}, \text{poles} = \{-100, -200\}}$$

9-5 Laplace of $f(t) = A((B+\alpha t)e^{-\alpha t})u(t)$, find zeros and poles!

$$F(s) = \mathcal{L}\{A(Be^{-\alpha t} + \alpha t e^{-\alpha t})u(t)\} = A\left(B\frac{1}{s+\alpha} + \alpha\frac{1}{(s+\alpha)^2}\right)$$

$$= \boxed{A\left(\frac{Bs + (B+1)\alpha}{(s+\alpha)^2}\right)} \rightarrow \boxed{\text{zeros} = \left\{-\left(\frac{B+1}{B}\right)\alpha, \infty\right\} \ \text{poles} = \{-\alpha, -\alpha\}}$$

9-9 Laplace of $f(t) = 10(3 - 5t - 2e^{-15t})u(t)$, find zeros & poles!

$$F(s) = \mathcal{L}\{10(3-5t-2e^{-15t})u(t)\} = 10\left(\frac{3}{s} - \frac{5}{s^2} - 2\frac{1}{(s+15)}\right)$$

$$= 10\left(\frac{3s(s+15) - 5(s+15) - 2s^2}{s^2(s+15)}\right) = 10\left(\frac{s^2 + 40s - 75}{s^2(s+15)}\right) = \boxed{10\left(\frac{(s-1.795)(s+41.795)}{s^2(s+15)}\right)}$$

$$\hookrightarrow \boxed{\text{zeros} = \{1.795, -41.795, \infty\}, \ \text{poles} = \{0, 0, -15\}}$$

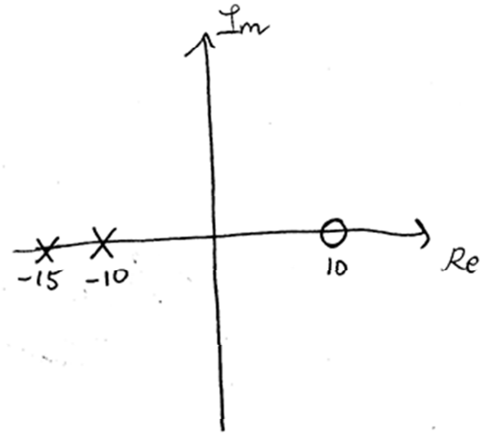
9-10 Find Laplace transforms and plot the pole-zero diagrams!

a) $f_1(t) = (25e^{-15t} - 20e^{-10t})\,u(t)$

$$F_1(s) = \mathcal{L}\{(25e^{-15t} - 20e^{-10t})u(t)\} = \frac{25}{s+15} - \frac{20}{s+10}$$

$$= \frac{25(s+10) - 20(s+15)}{(s+15)(s+10)} = \boxed{\frac{5(s-10)}{(s+15)(s+10)}}$$

$$\boxed{\text{poles} = \{-10, -15\},\ \text{zeros} = \{10, \infty\}}$$



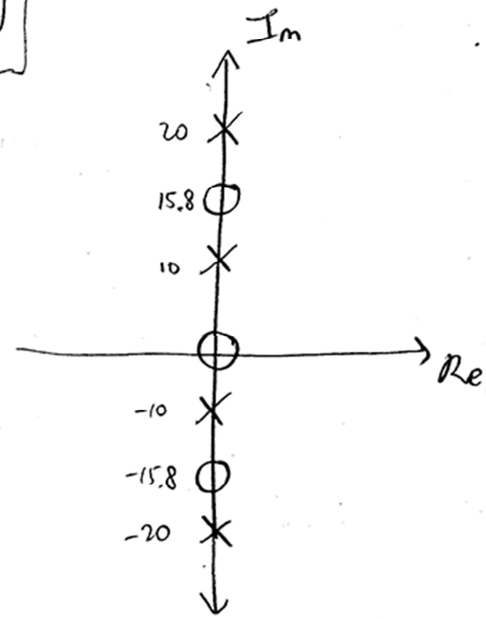
b) $f_2(t) = 10(\cos 10t + \cos 20t)\,u(t)$

$$F_2(s) = \mathcal{L}\{10(\cos 10t + \cos 20t)u(t)\} = 10\left(\frac{s}{s^2+10^2} + \frac{s}{s^2+20^2}\right)$$

$$= 10\left(\frac{s(s^2+20^2) + s(s^2+10^2)}{(s^2+10^2)(s^2+20^2)}\right)$$

$$= 10\left(\frac{s(2s^2+500)}{(s^2+10^2)(s^2+20^2)}\right) = \boxed{10\left(\frac{2s(s^2+250)}{(s^2+10^2)(s^2+20^2)}\right)}$$

$$\boxed{\begin{aligned}&\text{poles} = \{\pm 10j, \pm 20j\}\\ &\text{zeros} = \{0, \pm j\sqrt{250}, \infty\}\end{aligned}}$$



X = pole
O = zero

9-12 Find Laplace, locate poles & zeros of $F(s)$, verify with Matlab!

a) $f_1(t) = 5\delta(t) + 625\, t e^{-25t} u(t)$

$$F_1(s) = \mathcal{L}\{5\delta(t) + 625 t e^{-25t} u(t)\} = 5 + 625 \frac{1}{(s+25)^2} = \frac{5(s+25)^2 + 625}{(s+25)^2}$$

$$\boxed{= \frac{5(s^2+50s+750)}{(s+25)^2} = \frac{5(s-(-25+11.18j))(s-(-25-11.18j))}{(s+25)^2}}$$

$$\boxed{\text{Poles} = \{-25, -25\}, \quad \text{zeros} = \{-25 \pm 11.18j\}}$$

→ Can confirm in Matlab

b) $f_2(t) = (10 + 5e^{-10t}(\cos 10t + \sin 10t))u(t)$

$$F_2(s) = \mathcal{L}\{10 + 5(\cos 10t + \sin 10t)e^{-10t} u(t)\} = \frac{10}{s} + 5\left(\frac{(s+10)}{(s+10)^2+10^2} + \frac{10}{(s+10)^2+10^2}\right)$$

$$= \frac{10((s+10)^2+10^2) + 5(s(s+10) + 10s)}{s((s+10)^2+10^2)} = \frac{10(s^2+20s+200)+5(s^2+20s)}{s((s+10)^2+10^2)}$$

$$\boxed{= \frac{15(s^2+20s+\frac{400}{3})}{s((s+10)^2+10^2)} = \frac{15(s+10+5.77j)(s+10-5.77j)}{s(s+10+10j)(s+10-10j)}}$$

$$\boxed{\text{Poles} = \{0, -10 \pm 10j\}, \quad \text{zeros} = \{-10 \pm 5.77j, \infty\}}$$

9-23 Find inverse Laplace!

a) $F_1(s) = \dfrac{900}{(s+10)^2+30^2}$

$F_1(s) = A\dfrac{\beta}{(s+\alpha)^2+\beta^2}$ where $\left.\begin{matrix}\alpha=10\\ \beta=30\\ A\beta=900\end{matrix}\right\}$ $A=30$

$F_1(s) = 30\dfrac{30}{(s+10)^2+30^2}$ $\rightarrow$ $\boxed{f_1(t) = \mathcal{L}\{F_1(s)\} = 30e^{-10t}\sin 30t\ u(t)}$

b) $F_2(s) = \dfrac{3(s+10)}{(s+10)^2+30^2}$

$F_2(s) = B\dfrac{(s+\alpha)}{(s+\alpha)^2+\beta^2}$ where $\begin{matrix}\alpha=10\\ \beta=30\\ B=3\end{matrix}$

$\hookrightarrow$ $\boxed{f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = 3\cdot e^{-10t}\cos 30t\ u(t)}$

9-27 Find inverse Laplace and confirm w/ Matlab!

a) $F_1(s) = \dfrac{16s}{(s+3)(s^2+11s+10)} = \dfrac{16s}{(s+3)(s+1)(s+10)} = \dfrac{A}{s+3}+\dfrac{B}{s+1}+\dfrac{C}{s+10}$

$= \dfrac{A(s+1)(s+10)+B(s+3)(s+10)+C(s+1)(s+3)}{(s+3)(s+1)(s+10)}$ Solve for A, B, C!

$(A+B+C)s^2 + (11A+13B+4C) + (10A+30B+3C) = 16s$

$\begin{bmatrix}1&1&1\\11&13&4\\10&30&3\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix} = \begin{bmatrix}0\\16\\0\end{bmatrix} \rightarrow \begin{bmatrix}A\\B\\C\end{bmatrix} = \begin{bmatrix}1&1&1\\11&13&4\\10&30&3\end{bmatrix}^{-1}\begin{bmatrix}0\\16\\0\end{bmatrix} = \begin{bmatrix}3.429\\-0.889\\-2.540\end{bmatrix}$

Solve
w/ matlab

Then we have:

$$f_1(t) = \mathcal{L}^{-1}\left\{ \frac{3.429}{s+3} - \frac{0.889}{s+1} - \frac{2.540}{s+10} \right\}$$

$$\boxed{f_1(t) = \left(3.429\, e^{-3t} - 0.889\, e^{-t} - 2.540\, e^{-10t}\right) u(t)}$$

b) $F_2(s) = \dfrac{5(s^2+9)}{s(s^2+25)}$

$$F_2(s) = \frac{A}{s} + \frac{Bs+C}{s^2+25} = \frac{A(s^2+25) + Bs^2 + Cs}{s(s^2+25)} = \frac{(A+B)s^2 + Cs + 25A}{s(s^2+25)}$$

Solve for: $(A+B)s^2 + Cs + 25A = 5s^2 + 45$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 25 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 45 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1.8 \\ 3.2 \\ 0 \end{bmatrix}$$

So, we have:

$$f_2(t) = \mathcal{L}^{-1}\left\{ \frac{1.8}{s} + \frac{3.2\, s}{s^2+25} \right\} = \boxed{\left(1.8 + 3.2 \cos 5t\right) u(t)}$$

9-28 Find inverse transform!

a) $F_1(s) = \dfrac{(s+10^4)(s+10^5)}{s(s+10^3)(s+5\cdot 10^4)} = \dfrac{A}{s} + \dfrac{B}{s+10^3} + \dfrac{C}{s+5\cdot 10^4}$

$$= \frac{A(s+10^3)(s+5\cdot 10^4) + Bs(s+10^4\cdot 5) + Cs(s+10^3)}{s(s+10^3)(s+5\cdot 10^4)}$$

$$= \frac{(A+B+C)s^2 + (51000A + 50000B + 1000C)s + (5\cdot 10^7 A)}{s(s+10^3)(s+5\cdot 10^4)} = \frac{s^2 + 110000s + 10^9}{s(s+10^3)(s+5\cdot 10^4)}$$

Solve for

$$\begin{bmatrix} 1 & 1 & 1 \\ 51000 & 50000 & 1000 \\ 5\cdot 10^7 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 110000 \\ 10^9 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 20 \\ -18.184 \\ -0.816 \end{bmatrix}$$

So, we have:

$$f_1(t) = \mathcal{L}^{-1}\left\{ \frac{20}{s} - \frac{18.184}{s+10^3} - \frac{0.816}{s+5\cdot 10^4} \right\} = \boxed{\left(20 - 18.184 e^{-1000t} - 0.816\, e^{-50000t}\right) u(t)}$$

b) $F_2(s) = \dfrac{3(s^4+10s^2+4)}{s(s^2+1)(s^2+4)}$

$$F_2(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4} = \frac{A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)s(s^2+1)}{s(s^2+1)(s^2+4)}$$

$$= \frac{(A+B+D)s^4 + (C+E)s^3 + (5A+4B+D)s^2 + (4C+E)s + 4A}{s(s^2+1)(s^2+4)}$$

$$\hookrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 5 & 4 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 30 \\ 0 \\ 12 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ -5 \\ 0 \end{bmatrix}$$

We have: $f_2(t) = \mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{5s}{s^2+1} - \frac{5s}{s^2+4} \right\} = \boxed{(3 + 5\cos t - 5\cos 2t)\, u(t)}$

9-35 Find inverse transform!

a) $F_1(s) = \dfrac{e^{-5s}(s+20)}{(s+10)(s+30)} = e^{-5s}\left(\dfrac{A}{s+10} + \dfrac{B}{s+30}\right) = e^{-5s}\left(\dfrac{A(s+30) + B(s+10)}{(s+10)(s+30)}\right)$

Solve for: $(A+B)s + (30A + 10B) = s + 20$

$$\begin{bmatrix} 1 & 1 \\ 30 & 10 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

So, we have: $f_1(t) = \mathcal{L}^{-1}\left\{ e^{-5s}\left(\frac{0.5}{s+10} + \frac{0.5}{s+30} \right) \right\}$ → Delayed by 5

$$\boxed{f_1(t) = \left(0.5\, e^{-10(t-5)} + 0.5\, e^{-30(t-5)}\right) u(t-5)}$$

b) $F_2(s) = \dfrac{s e^{-5s} + 20}{(s+10)(s+30)}$

$$F_2(s) = e^{-5s}\frac{s}{(s+10)(s+30)} + \frac{20}{(s+10)(s+30)} = e^{-5s}\left(\frac{A}{s+10} + \frac{B}{s+30}\right) + \left(\frac{C}{s+10} + \frac{D}{s+30}\right)$$

$$= e^{-5s}\left(\frac{(A+B)s + (30A+10B)}{(s+10)(s+30)}\right) + \left(\frac{(C+D)s + (30C+10D)}{(s+10)(s+30)}\right)$$

Solve for $(A+B)s + (30A+10B) = s$

$(C+D)s + (30C+10D) = 20$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 30 & 10 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 30 & 10 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 20 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1.5 \\ 1.0 \\ -1.0 \end{bmatrix}$$

So, we have:

$$f_2(t) = \mathcal{L}^{-1}\left\{ e^{-5s}\left(\frac{-0.5}{s+10} + \frac{1.5}{s+30}\right) + \left(\frac{1}{s+10} - \frac{1}{s+30}\right) \right\}$$

$$\boxed{f_2(t) = \left(-0.5\, e^{-10(t-5)} + 1.5\, e^{-30(t-5)}\right) u(t-5) + \left(e^{-10t} - e^{-30t}\right) u(t)}$$

c) $F_3(s) = \dfrac{s + 20e^{-5s}}{(s+10)(s+30)}$

$$F_3(s) = \frac{s}{(s+10)(s+30)} + e^{-5s}\frac{20}{(s+10)(s+30)} = \left(\frac{A}{s+10} + \frac{B}{s+30}\right) + e^{-5s}\left(\frac{C}{s+10} + \frac{D}{s+10}\right)$$

$$= \frac{(A+B)s + (30A+10B) + e^{-5s}\left((C+D)s + (10C+30D)\right)}{(s+10)(s+30)}$$

Solve for :

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 30 & 10 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 30 & 10 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 20 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1.5 \\ 1 \\ -1 \end{bmatrix}$$

$$f_3(t) = \mathcal{L}^{-1}\left\{\frac{-0.5}{s+10} + \frac{1.5}{s+30} + e^{-5s}\left(\frac{1}{s+10} - \frac{1}{s+30}\right)\right\}$$

$$\boxed{f_3(t) = \left(-0.5e^{-10t} + 1.5e^{-30t}\right)u(t) + \left(e^{-10(t-5)} - e^{-30(t-5)}\right)u(t-5)}$$