# Frequency Response

We now know how to analyze and design ccts via s-domain methods which yield dynamical information

- Zero-state response
- Zero-input response
- Natural response
- Forced response

The responses are described by the exponential modes

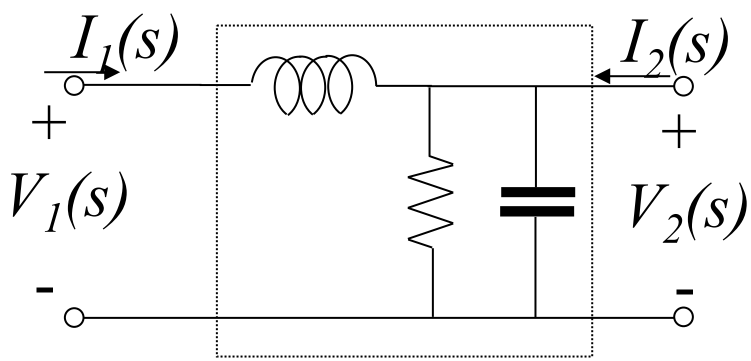
The modes are determined by the poles of the response Laplace Transform

We next will look at describing cct performance via frequency response methods

This guides us in specifying the cct pole and zero positions

# Transfer functions

Transfer function; measure input at one port, output at another




Inputs
Outputs

$$\text{Transfer function} = \frac{\text{zero - state response transform}}{\text{input signal transform}}$$

*(I.e., what the circuit does to your input)*

# Sinusoidal Steady-State Response

Consider a stable transfer function with a sinusoidal input $x(t) = A\cos(\omega t + \phi)$ $\quad X(s) = A\dfrac{s\cos(\phi) - \omega\sin(\phi)}{s^2 + \omega^2}$

The Laplace Transform of the response has poles

- Where the natural cct modes lie
  - These are in the open left half plane Re(s)<0
- At the input modes $s=+j\omega$ and $s=-j\omega$

Only the response due to the poles on the imaginary axis remains after a sufficiently long time

This is the sinusoidal steady-state response

# Sinusoidal Steady-State Response contd

**Input**

$$x(t) = A\cos(\omega t + \phi) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi$$

**Transform**

$$X(s) = A\cos\phi\frac{s}{s^2+\omega^2} - A\sin\phi\frac{\omega}{s^2+\omega^2}$$

**Response Transform**

$$Y(s) = T(s)X(s) = \frac{k}{s-j\omega} + \frac{k^*}{s+j\omega} + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots + \frac{k_N}{s-p_N}$$

**Response Signal**

$$y(t) = \underbrace{ke^{j\omega t} + k^*e^{-j\omega t}}_{\text{forced response}} + \underbrace{k_1e^{p_1t} + k_2e^{p_2t} + \ldots + k_Ne^{p_Nt}}_{\text{natural response}}$$

**Sinusoidal Steady State (SSS) Response**

$$y_{SS}(t) = ke^{j\omega t} + k^*e^{-j\omega t}$$

# Sinusoidal Steady-State Response contd

## Calculating the SSS response to $x(t) = A\cos(\omega t + \phi)$

### Residue calculation

$$k = \lim_{s \to j\omega} \left\lfloor (s - j\omega)Y(s) \right\rfloor = \lim_{s \to j\omega} \left\lfloor (s - j\omega)T(s)X(s) \right\rfloor$$

$$= \lim_{s \to j\omega} \left[ T(s)(s - j\omega)A \frac{s\cos\phi - \omega\sin\phi}{(s - j\omega)(s + j\omega)} \right] = T(j\omega)A\left[ \frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega} \right]$$

$$= \frac{1}{2}Ae^{j\phi}T(j\omega) = \frac{1}{2}A\left|T(j\omega)\right|e^{j(\phi + \angle T(j\omega))}$$

### Signal calculation

$$y_{ss}(t) = ke^{j\omega t} + k^{*}e^{-j\omega t}$$

$$= |k|e^{j\angle k}e^{j\omega t} + |k|e^{-j\angle k}e^{-j\omega t} = 2|k|\cos(\omega t + \angle k)$$

$$y_{ss}(t) = A|T(j\omega)|\cos(\omega t + \phi + \angle T(j\omega))$$

# Sinusoidal Steady-State Response contd

Response to $x(t) = A\cos(\omega t + \phi)$

is $y_{ss}(t) = A\,|T(j\omega)|\cos(\omega t + \phi + \angle T(j\omega))$

Output frequency = input frequency

Output amplitude = input amplitude $\times$ $|T(j\omega)|$

Output phase = input phase $+\angle T(j\omega)$

The Frequency Response of the transfer function T(s) is given by its evaluation as a function of a complex variable at $s=j\omega$

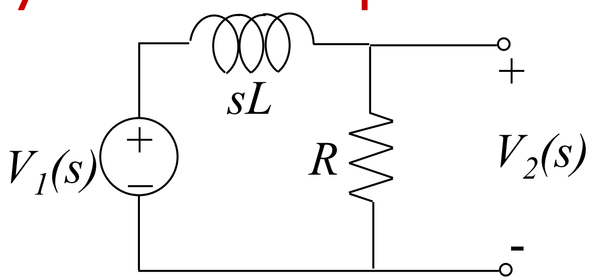
We speak of the amplitude response and of the phase response. They cannot independently be varied

$|T(j\omega)|$ *gain*

$\angle T(j\omega)$ *phase*

# Example 11-13, T&R 5th ed, p 527

Find the steady state output for $v_1(t)=A\cos(\omega t+\phi)$




Compute the s-domain transfer function T(s)

Voltage divider $T(s)=\dfrac{R}{sL+R}$

Compute the frequency response

$$|T(j\omega)|=\frac{R}{\sqrt{R^2+(\omega L)^2}},\qquad \angle T(j\omega)=-\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Compute the steady state output

$$v_{2SS}(t)=\frac{AR}{\sqrt{R^2+(\omega L)^2}}\cos\left[\omega t+\phi-\tan^{-1}(\omega L/R)\right]$$

# Terminology for Frequency Response

Based on shape of gain function of frequency

Passband: range of frequencies with nearly constant gain

Stopband: range of frequency with significantly reduced gain

Cutoff frequency: frequency associated with transition between bands

$$|T(j\omega_C)| = \frac{1}{\sqrt{2}} T_{\max}$$

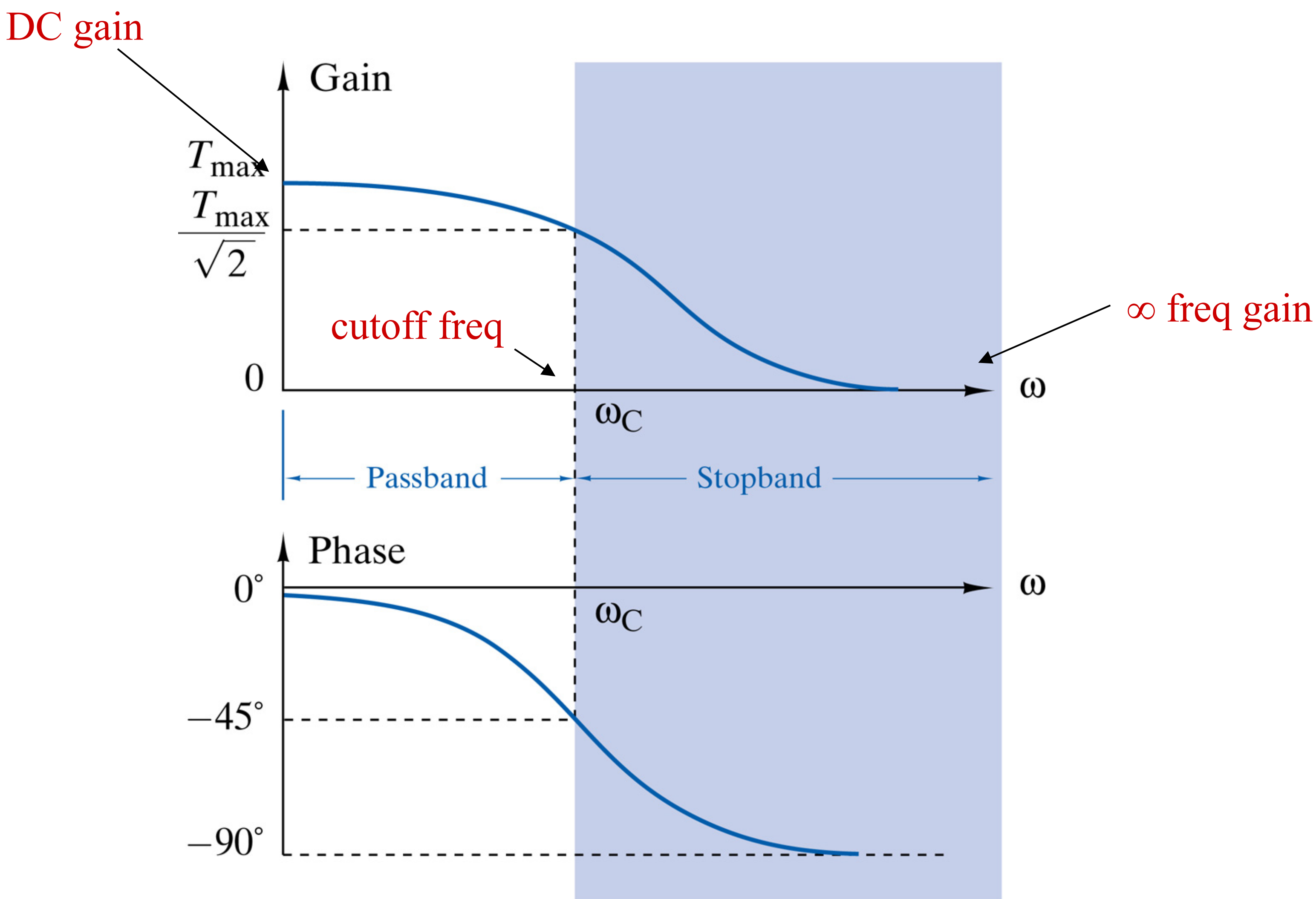
Low-pass filter: passband plus stopband

High-pass filter: stopband plus passband

Bandpass filter: one passband with two adjacent stopbands

Bandstop filter: one stopband with two adjacent passbands

# Terminology for Frequency Response




What kind of filter is this one?

# First-order low-pass filter

$$T(s) = \frac{K}{s+\alpha}$$

What is DC gain? What is ∞-freq gain? What is cutoff freq?
K, α real, α>0

First compute gain and phase

$$|T(j\omega)| = \frac{|K|}{\sqrt{\omega^2+\alpha^2}}$$

$$\angle T(j\omega) = \angle K - \arctan\left(\frac{\omega}{\alpha}\right)$$

DC gain

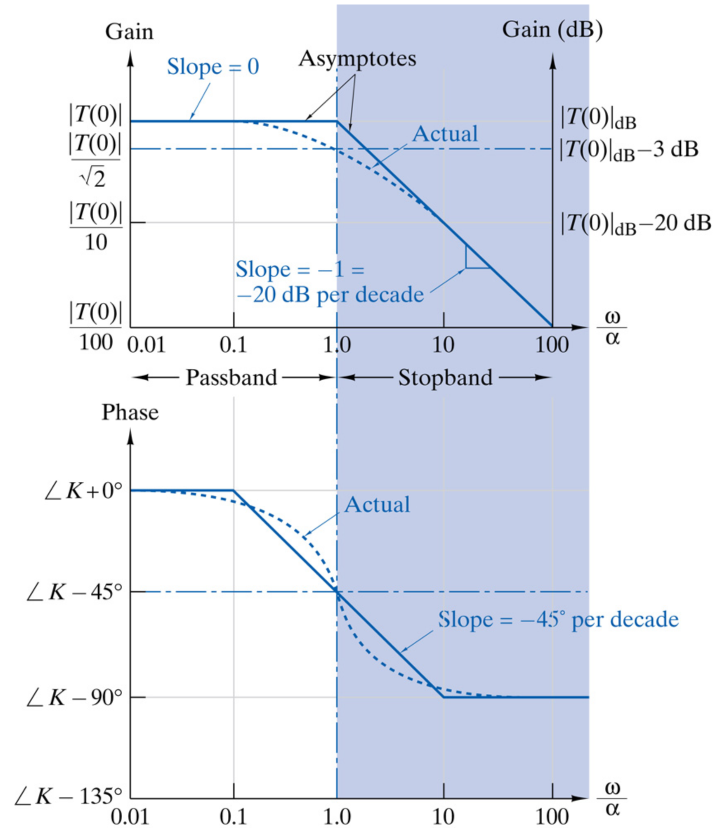
$$\lim_{\omega\to 0}|T(j\omega)| = \frac{|K|}{\alpha}$$

∞-freq

$$\lim_{\omega\to \infty}|T(j\omega)| = 0$$

Cutoff freq

$$|T(j\omega_c)| = \frac{1}{\sqrt{2}}T_{max} = \frac{1}{\sqrt{2}}\frac{|K|}{\alpha} \Rightarrow \omega_c = \alpha$$



# First-order high-pass filter

$$T(s)=\frac{Ks}{s+\alpha}$$

What is DC gain? What is ∞-freq gain? What is cutoff freq?
K, α real, α>0

First compute gain and phase

$$\left|T(j\omega)\right|=\frac{\left|K\right|\omega}{\sqrt{\omega^2+\alpha^2}}$$

$$\angle T(j\omega)=\angle K+90-\arctan\left(\frac{\omega}{\alpha}\right)$$

DC gain

$$\lim_{\omega\to 0}\left|T(j\omega)\right|=0$$

∞-freq

$$\lim_{\omega\to\infty}\left|T(j\omega)\right|=\left|K\right|$$

Cutoff freq

$$\left|T(j\omega_c)\right|=\frac{1}{\sqrt{2}}T_{\max}=\frac{1}{\sqrt{2}}\left|K\right|\Rightarrow\omega_c=\alpha$$

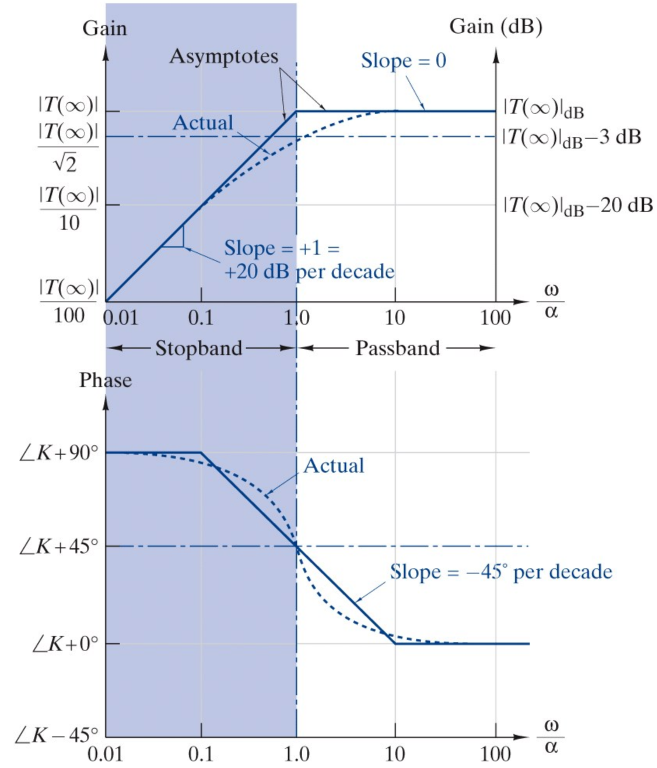



Figure 12-10

# Bandpass filter

$$T(s) = T_1(s) \times T_2(s) = \underbrace{\left(\frac{K_1 s}{s+\alpha_1}\right)}_{high-pass}\underbrace{\left(\frac{K_2}{s+\alpha_2}\right)}_{low-pass} \qquad (\alpha_1 << \alpha_2)$$

Gain function

$$|T(j\omega)| = \frac{|K_1|\omega}{\sqrt{\omega^2+\alpha_1^2}}\frac{|K_2|}{\sqrt{\omega^2+\alpha_2^2}}$$

$\text{Low}(\omega << \alpha_1 << \alpha_2)\text{freq}$

$$|T(j\omega)| \approx \frac{|K_1||K_2|\omega}{\alpha_1\alpha_2}$$

$\text{Mid}(\alpha_1 << \omega << \alpha_2)\text{freq}$

$$|T(j\omega)| \approx \frac{|K_1||K_2|}{\alpha_2}$$

$\text{High}(\alpha_1 << \alpha_2 << \omega)\text{freq}$

$$|T(j\omega)| \approx \frac{|K_1||K_2|}{\omega}$$

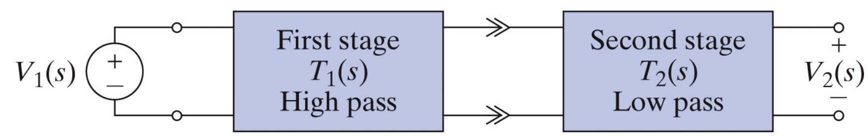



Figure 12-17



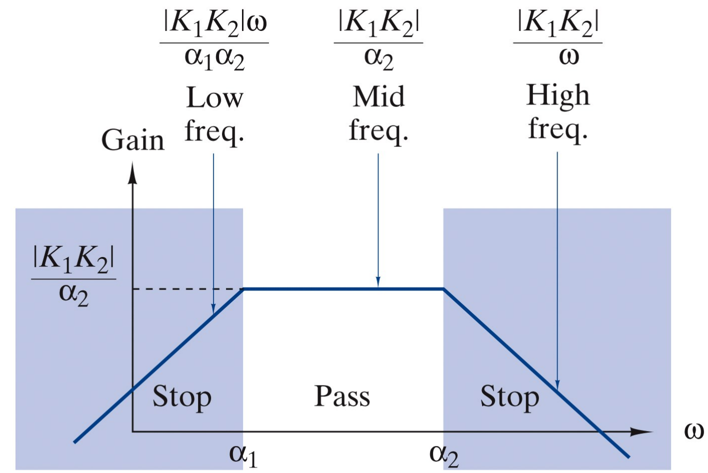
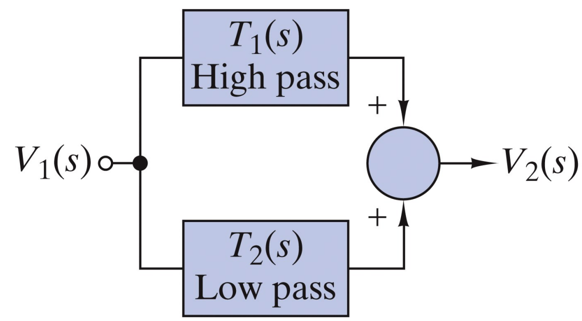



Figure 12-18

# Bandstop filter

$$T(s) = T_1(s) + T_2(s) = \underbrace{\left(\frac{K_1 s}{s+\alpha_1}\right)}_{high-pass} + \underbrace{\left(\frac{K_2}{s+\alpha_2}\right)}_{low-pass} \qquad (\alpha_2 << \alpha_1)$$

$$|T_1(j\omega)| = \frac{|K_1|\omega}{\sqrt{\omega^2+\alpha_1^2}} \qquad |T_2(j\omega)| = \frac{|K_2|}{\sqrt{\omega^2+\alpha_2^2}}$$






Low$(\omega << \alpha_2 << \alpha_1)$freq

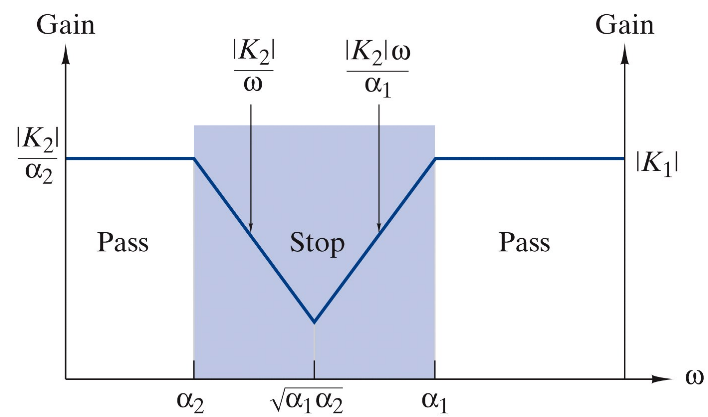
$$|T(j\omega)| \approx \frac{|K_2|}{\alpha_2}$$

Mid$(\alpha_2 << \omega << \alpha_1)$freq

$$|T(j\omega)| \approx \frac{|K_1|\omega}{\alpha_1} + \frac{|K_2|}{\omega}$$

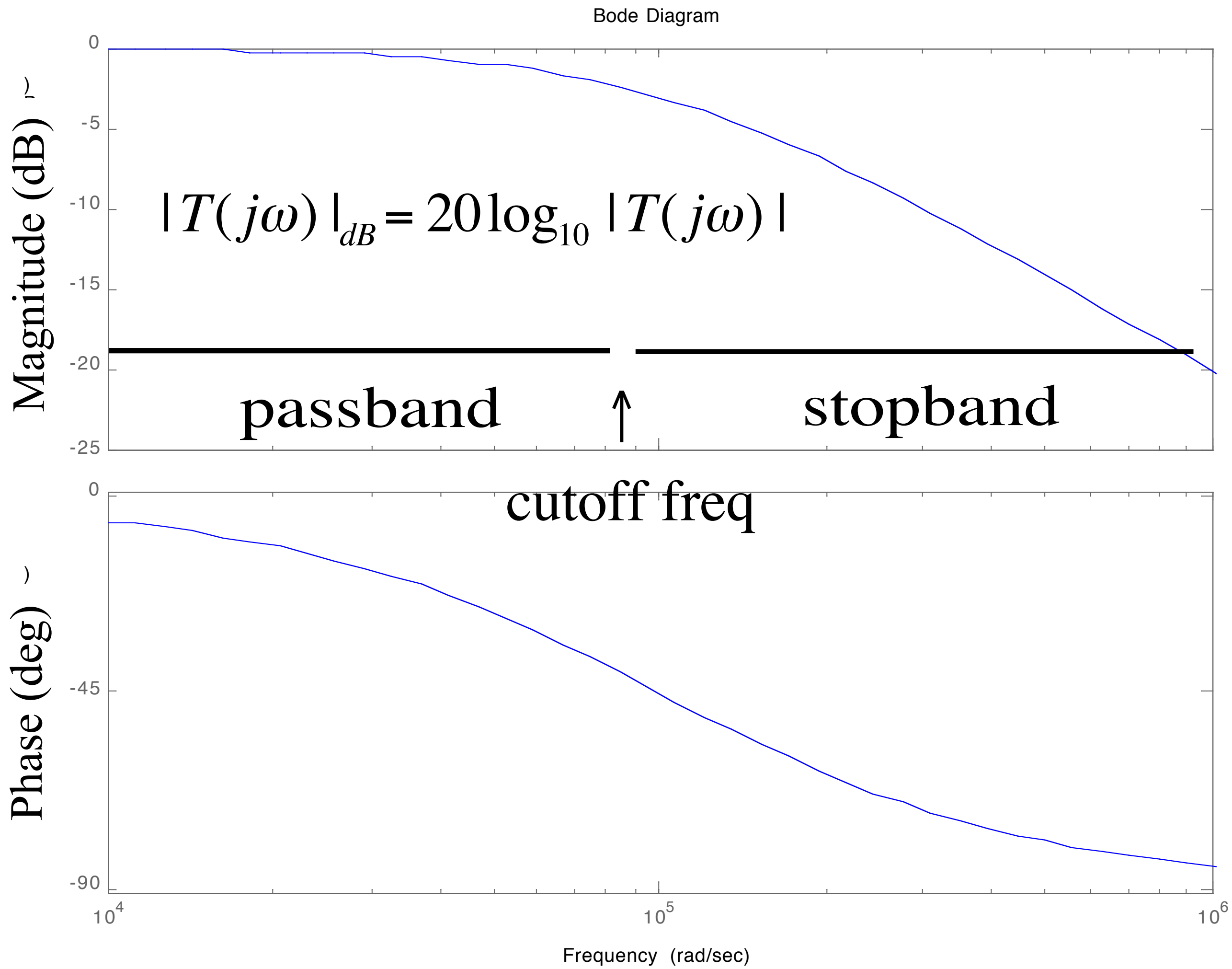
High$(\alpha_2 << \alpha_1 << \omega)$freq

$$|T(j\omega)| \approx |K_1|$$



# Frequency Response – Bode Diagrams

## Log-log plot of mag(T), log-linear plot arg(T) versus ω



# Matlab Commands for Bode Diagram

Specify component values

Set up transfer function

```
>> R=1000;L=0.01;
>> Z=tf(R,[L R])

Transfer function:
    1000
-------------
0.01 s + 1000

>> bode(Z)
```

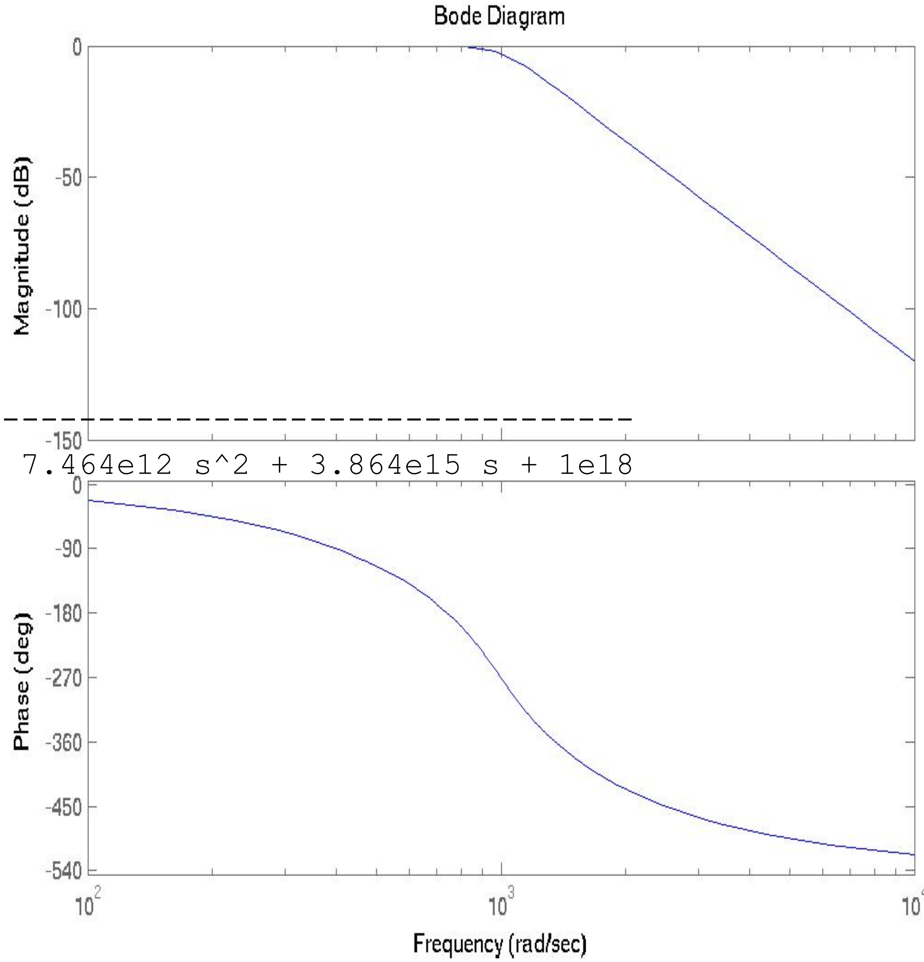
# Frequency Response Descriptors

## Lowpass Filters

```
[num,den]=butter(6,1000,'s');
lpass=tf(num,den);
lpass

Transfer function:
                                          1e18
--------------------------------------------------------------------------------
s^6 + 3864 s^5 + 7.464e06 s^4 + 9.142e09 s^3 + 7.464e12 s^2 + 3.864e15 s + 1e18

bode(lpass)
```

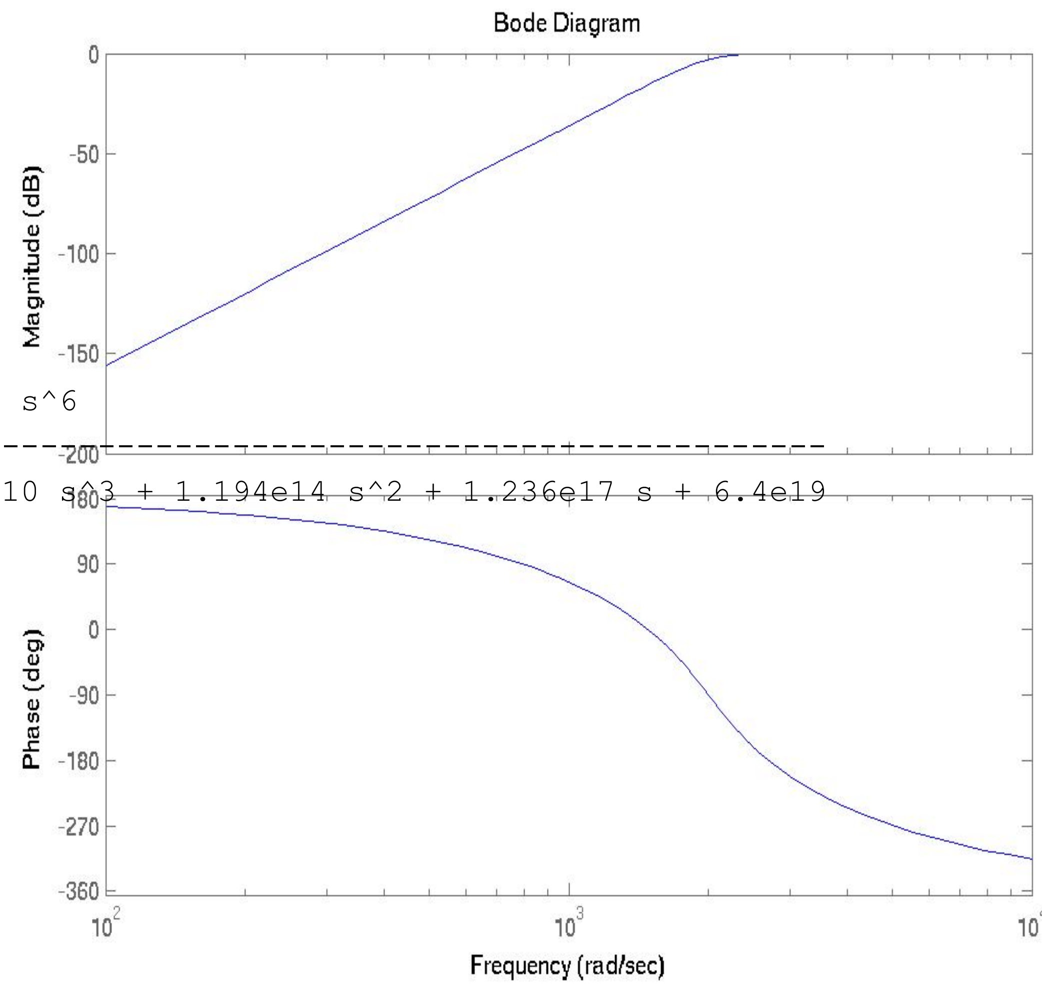


# High Pass Filters

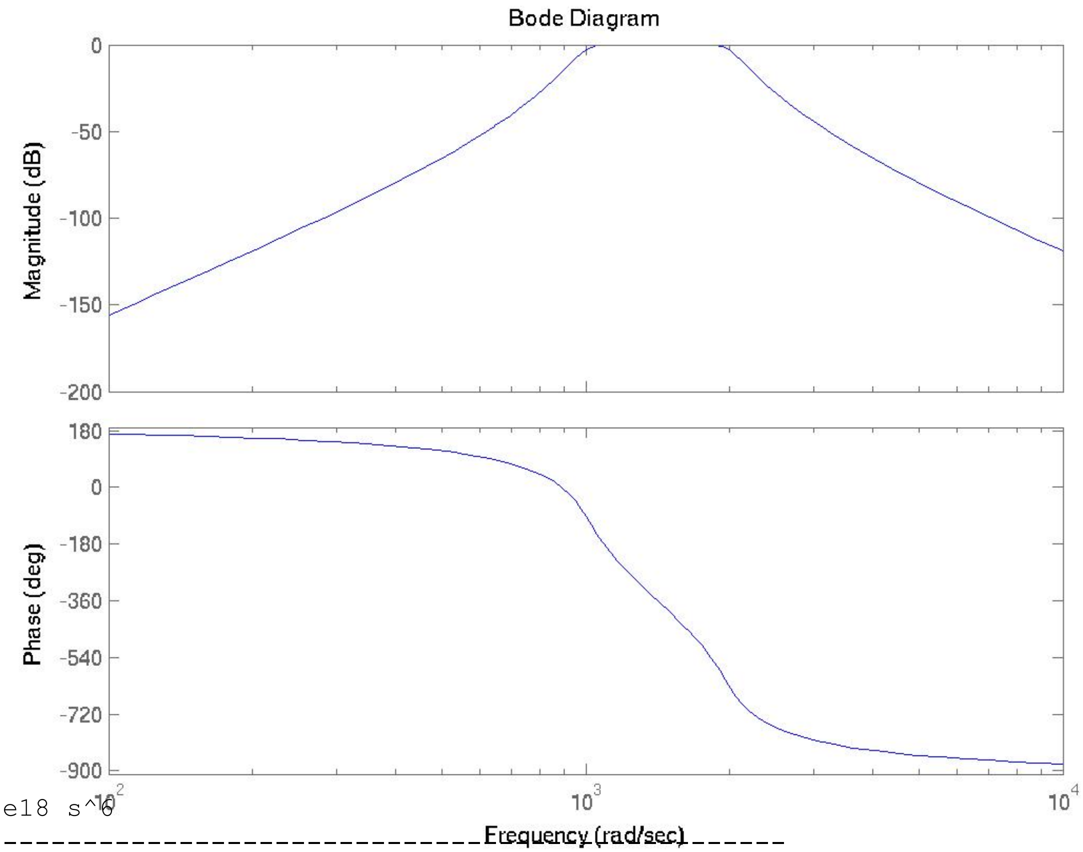
```
[num,den]=butter(6,2000,'high','s');
hpass=tf(num,den)

Transfer function:
                                          s^6
-------------------------------------------------------------------------------------
s^6 + 7727 s^5 + 2.986e07 s^4 + 7.313e10 s^3 + 1.194e14 s^2 + 1.236e17 s + 6.4e19

bode(hpass)
```



# Bandpass Filters



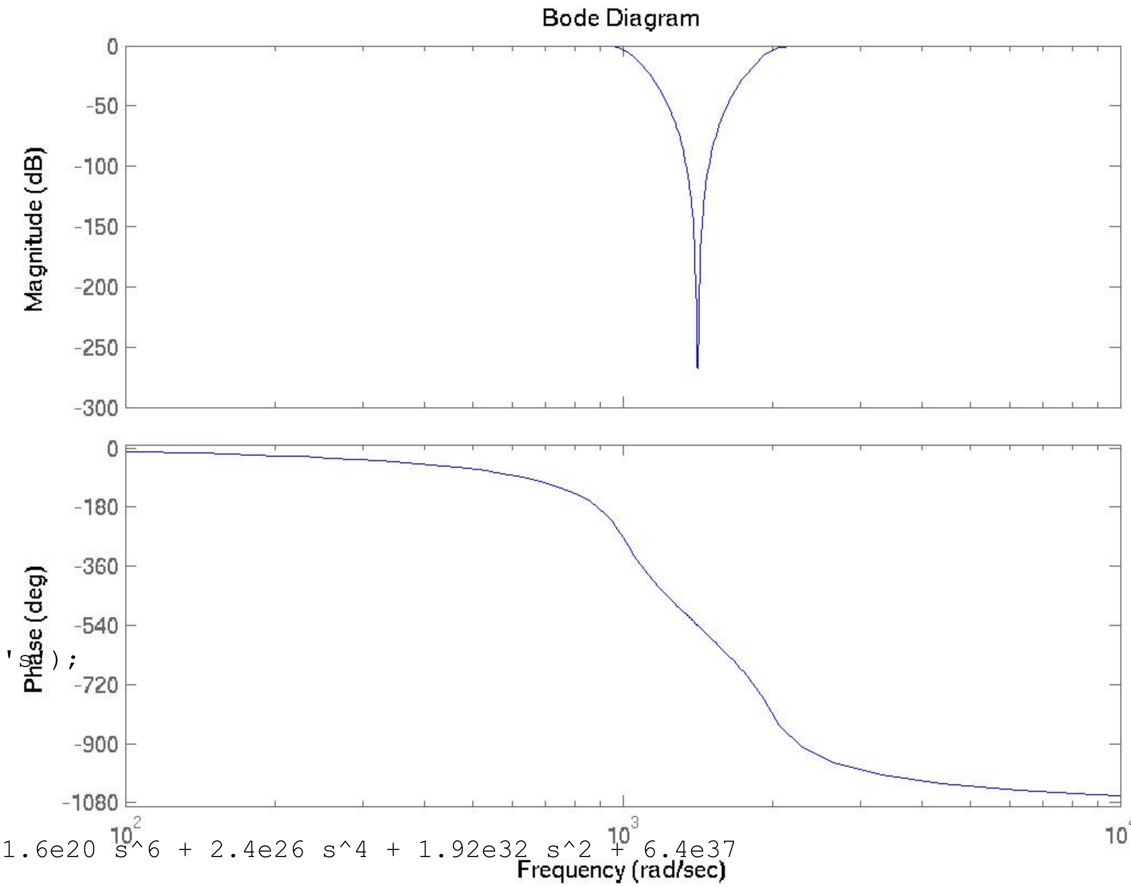

```
[num,den]=butter(6,[1000 2000],'s');
bpass=tf(num,den)

Transfer function:
                                       1e18 s^6
------------------------------------------------------------------------------------
s^12 + 3864 s^11 + 1.946e07 s^10 + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6

      + 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37


bode(bpass)
```

# Bandstop Filters




```
[num,den]=butter(6,[1000 2000],'stop','s');
bstop=tf(num,den)

Transfer function:

      s^12 + 1.2e07 s^10 + 6e13 s^8 + 1.6e20 s^6 + 2.4e26 s^4 + 1.92e32 s^2 + 6.4e37
-------------------------------------------------------------------------------------------
s^12 + 3864 s^11 + 1.946e07 s^10 + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6

        + 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37


bode(bstop)
```