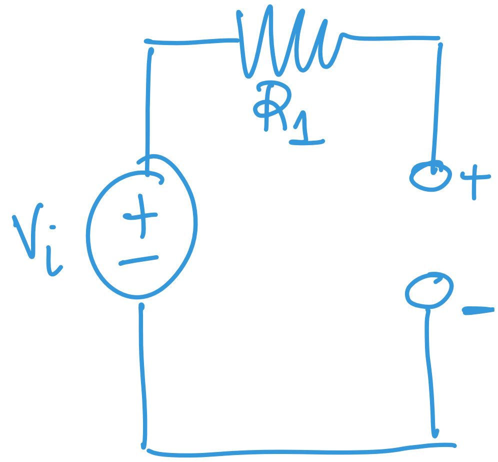
1._ Part I

Under DC excitations, we know the capacitor behaves as an open circuit. Therefore, we have

[+1 point]



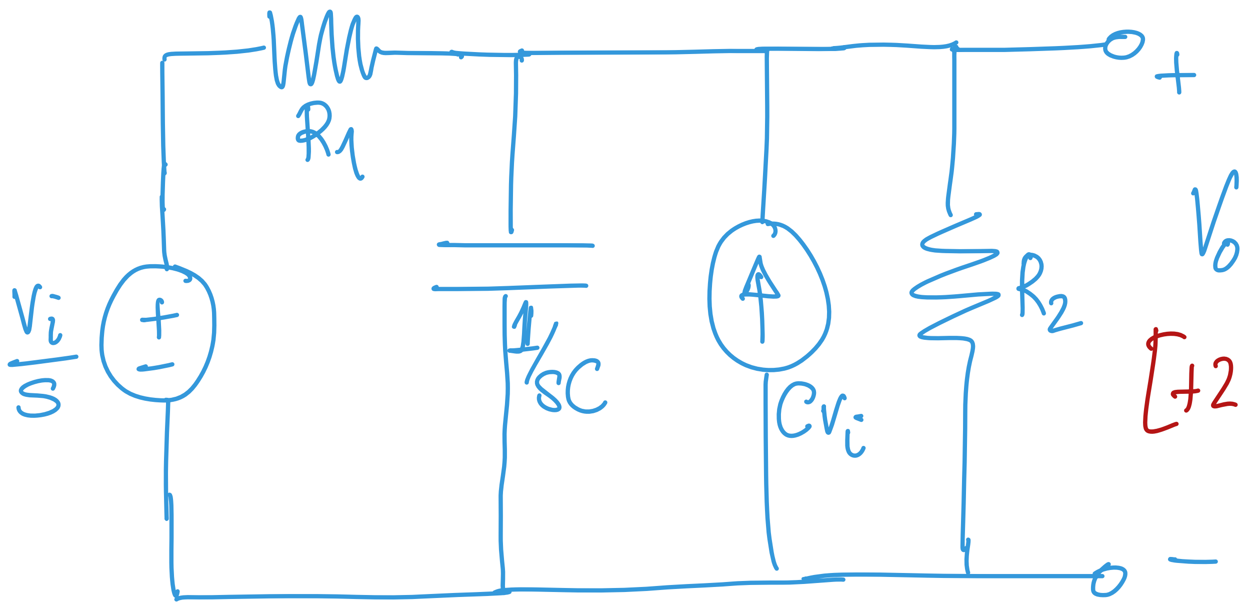
From the picture, we deduce that

$V_c(0) = V_i$

[+1 point]

Part II

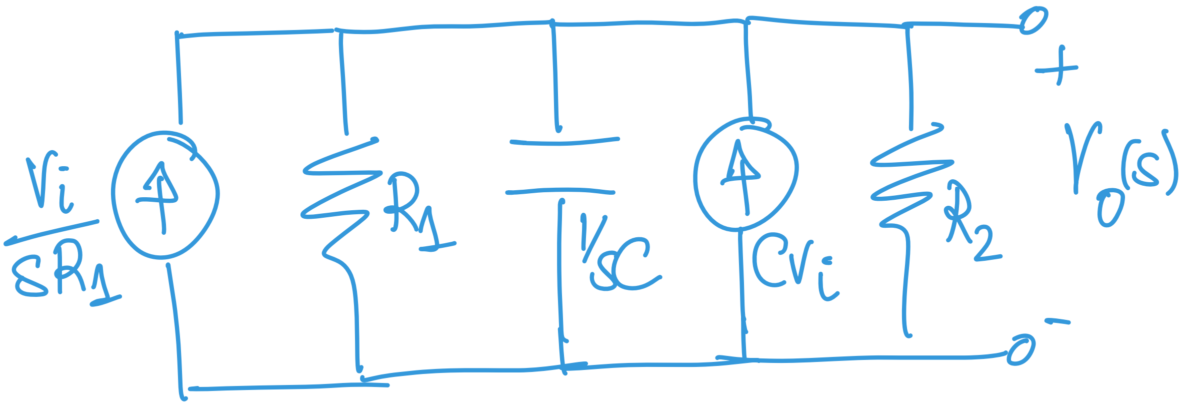
Next, we transform the circuit into the s-domain using a current source to account for the initial condition.



[+2 points]

Part III

We use source transformation to redraw the circuit as



[+1 point]

We can now use current division to find the current through the R_2-impedance as

$$i_{R_2} = \frac{1/R_2}{\frac{1}{R_2} + \frac{1}{R_1} + sC}\left(\frac{V_i}{sR_1} + CV_i\right)$$

Finally,

$$V_o(s) = R_2 \cdot i_{R_2} = \frac{R_2 R_1}{R_1 + R_2 + R_1 R_2 Cs} \cdot \frac{V_i + CR_1 V_i s}{sR_1}$$

$$= \frac{R_2 (1 + CR_1 s) V_i}{(R_1 + R_2 + R_1 R_2 Cs)\, s}$$

[+1 point]

Part IV

Zero-state component corresponds to when the initial conditions are set to zero. Therefore,

$$(V_0)_{zs}(s) = \frac{R_2 V_i}{(R_1+R_2+R_1R_2Cs)s} \qquad [+1\text{ point}]$$

Zero-input component corresponds to when the input is set to zero. Therefore,

$$(V_0)_{zi}(s) = \frac{R_1R_2CV_i}{R_1+R_2+R_1R_2Cs} \qquad [+1\text{ point}]$$

Part V

Forced component corresponds to the poles of the input. In this case, the input has a pole at 0. To answer the question, we use <u>partial fractions to decompose $V_0(s)$</u> as

$$V_0(s) = \frac{A}{s} + \frac{B}{s+\frac{R_1+R_2}{R_1R_2C}}$$

We use the residue method to find A & B.

$$A = \lim_{s \to 0} s\,V_0(s) = \lim_{s \to 0} \frac{R_2(1 + CR_1 s)V_i}{R_1 + R_2 + R_1 R_2 C s} =$$

$$= \frac{R_2 V_i}{R_1 + R_2}$$

$$B = \lim_{s \to -\frac{R_1+R_2}{R_1R_2C}} \left(s + \frac{R_1+R_2}{R_1R_2C}\right) V_0(s) =$$

$$= \lim_{s \to -\frac{R_1+R_2}{R_1R_2C}} \frac{R_2(1 + CR_1 s)V_i}{s \cdot CR_1R_2} =$$

$$= \frac{R_2 V_i \left(1 - \frac{\cancel{C}\cancel{R_1}(R_1+R_2)}{\cancel{C}\cancel{R_1}R_2}\right)}{-\frac{(R_1+R_2)}{\cancel{R_1R_2C}}\,\cancel{R_1R_2C}} = \frac{\cancel{R_2}V_i\left(+\frac{R_1}{\cancel{R_2}}\right)}{+(R_1+R_2)} =$$

$$= \frac{R_1 V_i}{R_1 + R_2}$$

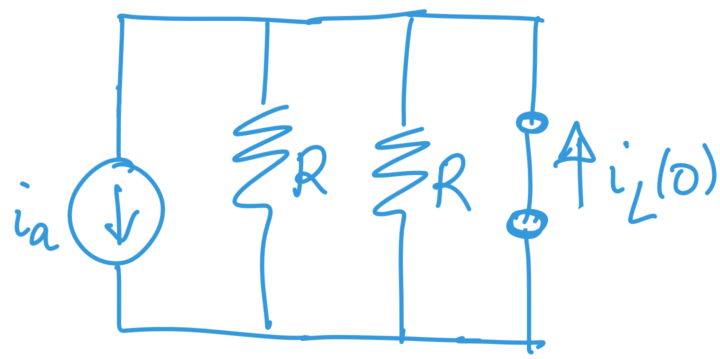
Therefore, we conclude that

$$(V_0)_{fr}(s) = \frac{R_2 v_i}{(R_1+R_2)s}$$ [+1 point]

$$(V_0)_{nr}(s) = \frac{R_1 v_i}{R_1+R_2} \cdot \frac{1}{s+\frac{R_1+R_2}{R_1 R_2 C}}$$ [+1 point]

2.– **Part I**

Under DC excitation, we know the inductor behaves as a short circuit. Therefore, we have



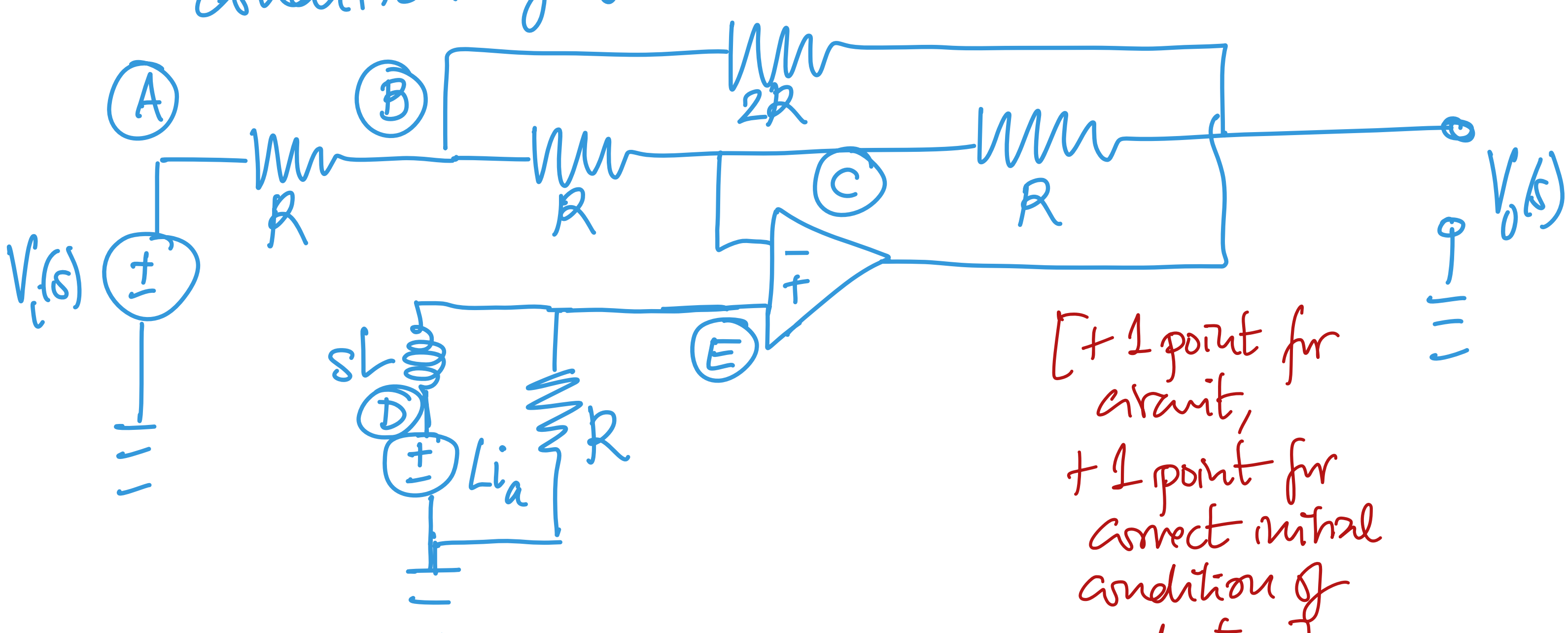
From the picture we deduce that

$$i_L(0) = i_a$$

[+1 point]

Part II

We transform the circuit into the S-domain, using a voltage source to represent the initial condition of the inductor.



[+1 point for circuit,
+1 point for correct initial condition of inductor]

Part III

We use the node labels provided in the plot above. Right away, we have that

$$V_A = V_i(s) \quad ; \quad V_D = Li_a$$ [+1 point]

Because of ideal op-amp conditions, we know that the op-amp has infinite input impedance, and hence R and sL are in series with the voltage source Li_a. Therefore,

$$V_E = \frac{R}{R+sL} Li_a$$ [+1 point]

KCL @ node B

$$\frac{1}{R}(V_B - V_A) + \frac{1}{2R}(V_B - V_0) + \frac{1}{R}(V_B - V_C) = 0$$ [+1 point]

KCL @ node C

$$\frac{1}{R}(V_C - V_B) + \frac{1}{R}(V_C - V_0) = 0$$ [+1 point]

Finally, we do not write KCL for the output node of the op-amp, relying instead on ideal op-amp conditions.

$$V_C = V_E$$ [+1 point]

Therefore, we have

$$V_0 = 2V_C - V_B = \frac{2RLi_a}{R+sL} - V_B$$

Substituting into KCL for Ⓑ,

$$\frac{1}{R}(V_B - V_i) + \frac{1}{2R}\left(V_B + V_B - \frac{2RLi_a}{R+sL}\right) + \frac{1}{R}\left(V_B - \frac{RLi_a}{R+sL}\right) = 0$$

$$V_B - V_i + V_B - \frac{RLi_a}{R+sL} + V_B - \frac{RLi_a}{R+sL} = 0$$

Hence

$$V_B = \frac{1}{3}\left(V_i + \frac{2RLi_a}{R+sL}\right)$$

And therefore

$$V_0(s) = \frac{2RLi_a}{R+sL} - \frac{1}{3}V_i(s) - \frac{2}{3}\frac{RLi_a}{R+sL} =$$

$$= -\frac{1}{3}V_i(s) + \frac{4}{3}\frac{RLi_a}{R+sL}$$

Part IV

Substituting the values provided, we get

$$V_0(s) = -\frac{1}{3} V_i(s) + \frac{4}{3} \cdot \frac{10^{2} \cdot 10^{-3} \cdot 10^{-2}}{10^{2} + s\,10^{-3}} =$$

$$= -\frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{4}{3}\,\frac{1}{10^{5}+s} =$$

$$= -\frac{1}{3}\,\frac{1}{s} + \frac{1}{3}\,\frac{1}{s+1} + \frac{4}{3}\,\frac{1}{s+10^{5}}$$

[+1 point]

Taking inverse Laplace transforms, we get

$$v_0(t) = \left(-\frac{1}{3} + \frac{1}{3} e^{-t} + \frac{4}{3} e^{-10^{5} t}\right) u(t)$$

$$= -\frac{1}{3}\left(1 - e^{-t} - 4 e^{-10^{5} t}\right) u(t)$$

[+1 point]

3.- **Part I**

To compute the gain and phase functions, we evaluate the transfer function at $s = j\omega$,

$$T(j\omega) = \frac{10 j\omega}{-\omega^2 + 11 j\omega + 10}$$

Therefore

$$|T(j\omega)| = \frac{10\omega}{\sqrt{(10-\omega^2)^2 + 121\omega^2}} = \frac{10\omega}{\sqrt{100 + \omega^4 + 101\omega^2}}$$

[+1 point]

$$\angle T(j\omega) = \frac{\pi}{2} - \arctan \frac{11\omega}{10-\omega^2}$$ [+1 point]

Part II

$$|T(j0)| = \frac{0}{\sqrt{100}} = 0$$ [+0.5 point]

$$|T(j\infty)| = \lim_{\omega \to \infty} |T(j\omega)| = 0$$ [+0.5 point]

$$\angle T(j0) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \text{ rad}$$ [+0.5 point]

$$\angle T(j\infty) = \frac{\pi}{2} - \pi = -\frac{\pi}{2} \text{ rad}$$ [+0.5 point]

To compute the cutoff frequencies, we have to compute the maximum value of the gain function. Note that $|T(j\omega)|$ and $|T(j\omega)|^2$ achieve the maximum at the same point, so we use the latter (which is easier to derive),

$$\frac{100w^2}{100 + w^4 + 101w^2}$$

We derive it

$$\frac{200w(100 + w^4 + 101w^2) - 100w^2(4w^3 + 202w)}{(100 + w^4 + 101w^2)^2}$$

This vanishes when the numerator vanishes, i.e.,

$$0 = -200w^5 + 20000w = 200w(-w^4 + 100)$$

i.e., $\omega = 0$ or $\omega^4 = 100$

$$\omega^2 = \pm 10$$

$$\omega = \sqrt{10}$$

So the maximum is

$$T_{max} = \frac{10 \cdot \sqrt{10}}{\sqrt{100 + 100 + 101 \cdot 10}} = \frac{10\sqrt{10}}{11\sqrt{10}} = \boxed{\frac{10}{11}}$$

[+1 point]

Therefore, the cutoff frequencies are found by solving:

$$\frac{10\omega_c}{\sqrt{100 + \omega_c^4 + 101\omega_c^2}} = \boxed{|T(j\omega_c)| = \frac{T_{max}}{\sqrt{2}}} = \frac{10}{11\sqrt{2}}$$

$$\frac{\cancel{100}\omega_c^2}{100 + \omega_c^4 + 101\omega_c^2} = \frac{\cancel{100}}{121 \cdot 2}$$

$$242\omega_c^2 = 100 + \omega_c^4 + 101\omega_c^2$$

$$100 + \omega_c^4 - 141\omega_c^2 = 0 \qquad \omega_c^2 = z$$

$$z^2 - 141z + 100 = 0$$

$$z = \frac{141 \pm \sqrt{141^2 - 4 \cdot 100}}{2} \simeq \begin{cases} 0.7128 \\ 140.287 \end{cases}$$

Therefore

$$w_c = \sqrt{z} = \begin{cases} 0.8442 \\ 11.843 \end{cases}$$

[+1 point]

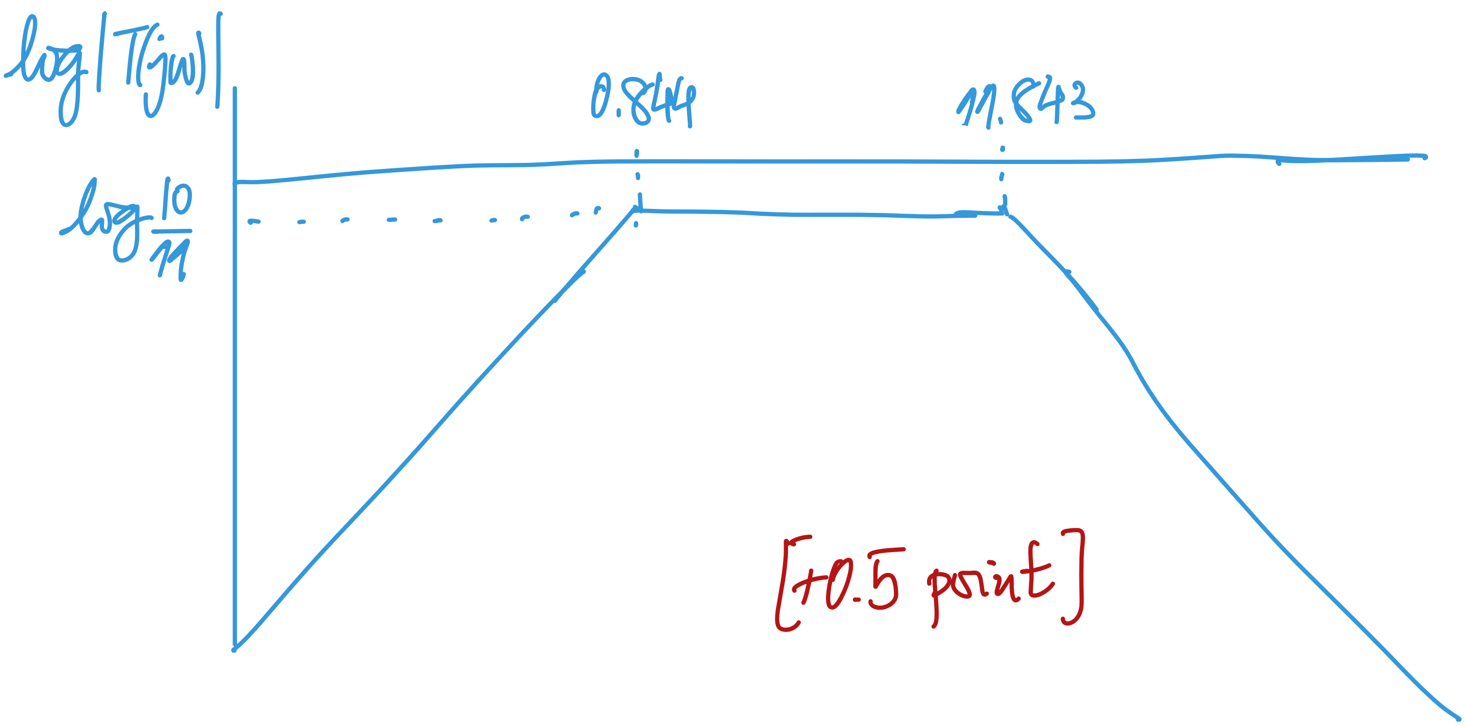
Part III

$$|T(j\omega)| = \frac{10\omega}{\sqrt{100 + \omega^4 + 101\omega^2}}$$

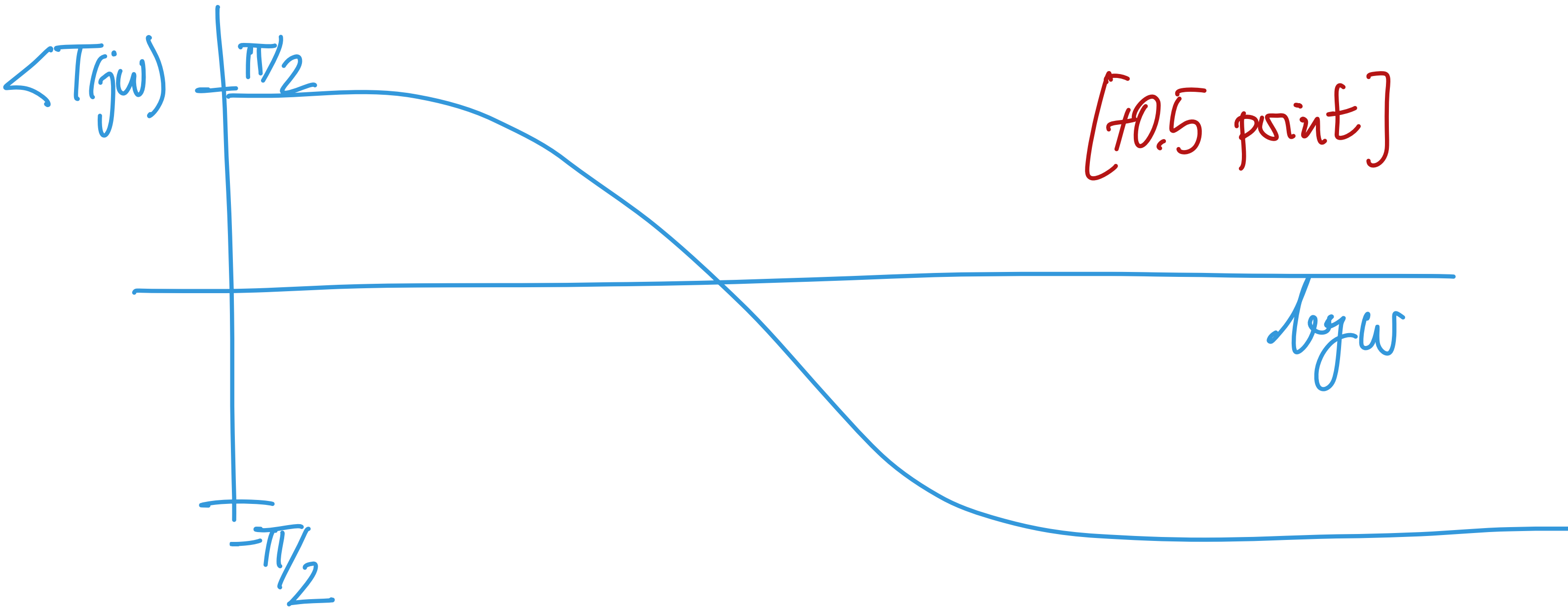
For $\omega \ll 1$, $|T(j\omega)| \approx \dfrac{10\omega}{10} = \omega$

For $\omega \gg 1$, $|T(j\omega)| \approx \dfrac{10\omega}{\omega^2} = \dfrac{10}{\omega}$

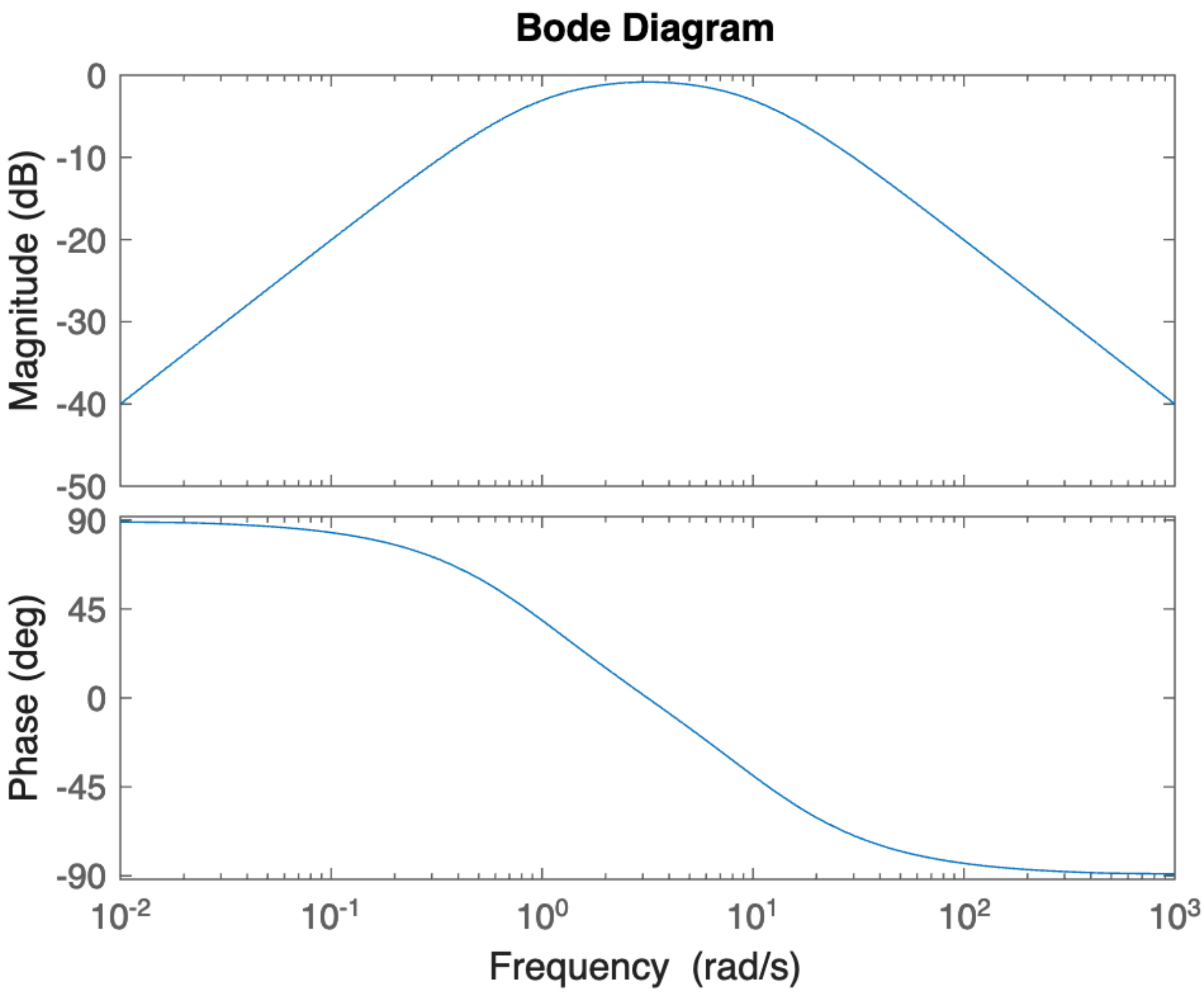
Sketches can be done in linear or logarithmic scale.



[+0.5 point]



[+0.5 point]



This is a bandpass filter. [+1 point]

Part IV

The input is

$$v_{in}(t) = 0.33\cos\left(\frac{t}{6} - \pi\right) + \cos(2t) + 0.25\cos\left(30t + \frac{\pi}{3}\right)$$

We therefore compute

$$\left|T\left(j\tfrac{1}{6}\right)\right| \simeq 0.164 \qquad \angle T\left(j\tfrac{1}{6}\right) \simeq 1.389 \text{ rad}$$

$$|T(j2)| \simeq 0.877 \qquad \angle T(j2) = 0.266 \text{ rad}$$

$$|T(j30)| \simeq 0.31 \qquad \angle T(j30) = -1.21 \text{ rad}$$

Therefore

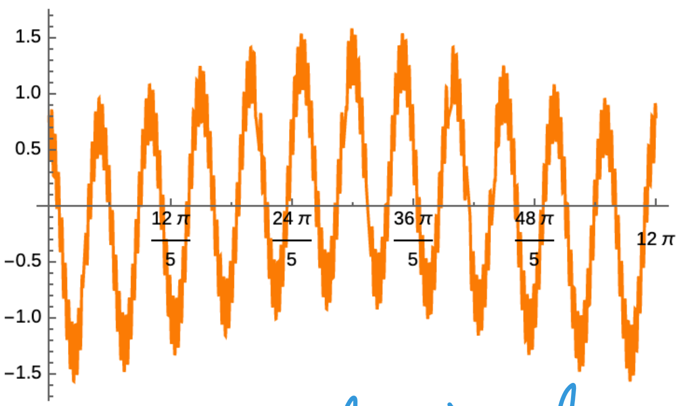
$$v_o^{ss}(t) = 0.055\cos\left(\frac{t}{6} - 1.75\right) + 0.877\cos(2t + 0.266)$$

$$+ 0.079\cos(30t - 0.168)$$

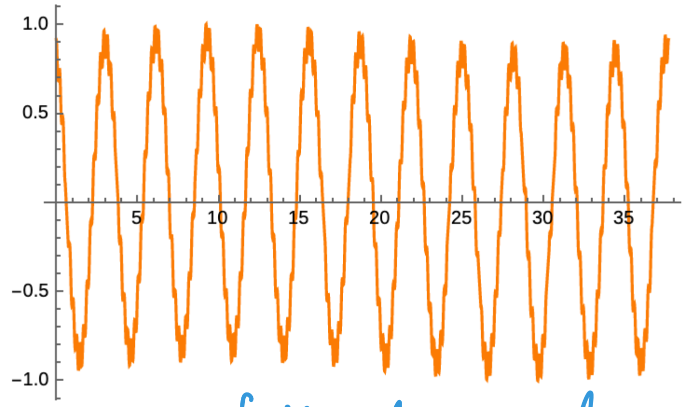
[+1 point]

Yes, the transfer function reasonably accomplished the engineer's goals. The time-varying bias almost disappeared in the filtered signal, and the noise was significantly reduced. Meanwhile, the signal with frequency

2 rad/s remained strong. [+1 point]



original signal
$v_{in}(t)$



filtered signal
$v_o^{ss}(t)$

4.- **Part I**

We find the poles of the transfer function.

$$s^2+11s+10=0$$

$$s=\frac{-11\pm\sqrt{121-4\cdot 10}}{2}=\frac{-11\pm 9}{2}=\begin{cases}-1\\-10\end{cases}$$

[+1 point]

Therefore

$$T(s)=\frac{10s}{s^2+11s+10}=\frac{10s}{(s+1)(s+10)}=\frac{s}{s+1}\cdot\frac{10}{s+10}$$

$\alpha_1=1$ [+1 point]

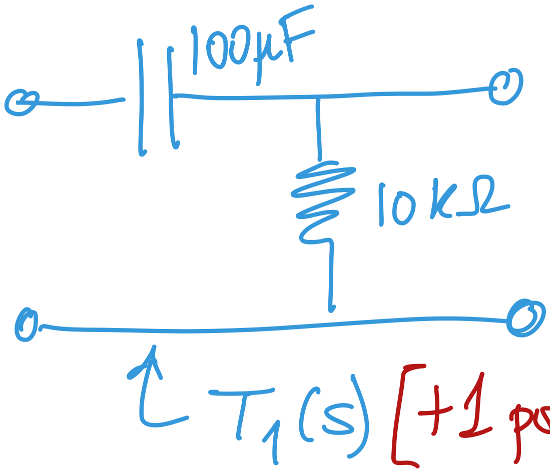
$\alpha_2=10$

Part II

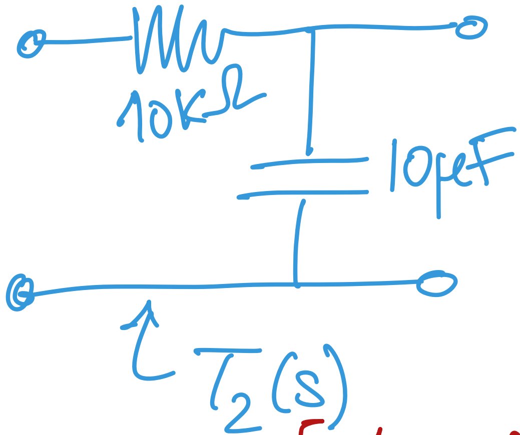
$$T_1(s)=\frac{s}{s+1}=\frac{1}{1+\frac{1}{s}}=\frac{10^4}{10^4+\frac{1}{s10^{-4}}}$$ [+1 point]

$$T_2(s)=\frac{10}{s+10}=\frac{10/s}{1+10/s}=\frac{1/s10^{-5}}{10^4+\frac{1}{s10^{-5}}}$$ [+1 point]

We have manipulated the numerator/denominator of these transfer functions to make sure we can design circuits that do not employ inductors. We design them next as voltage dividers.



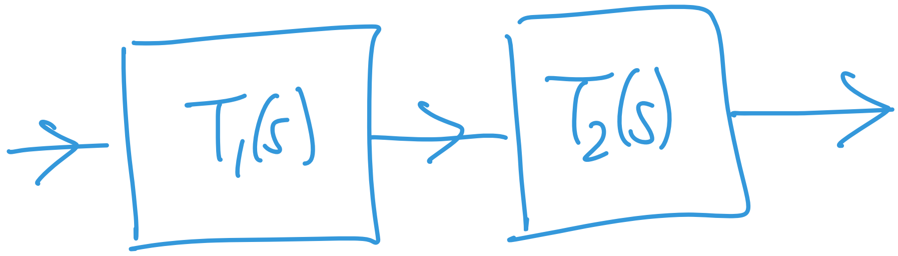
[+1 point]

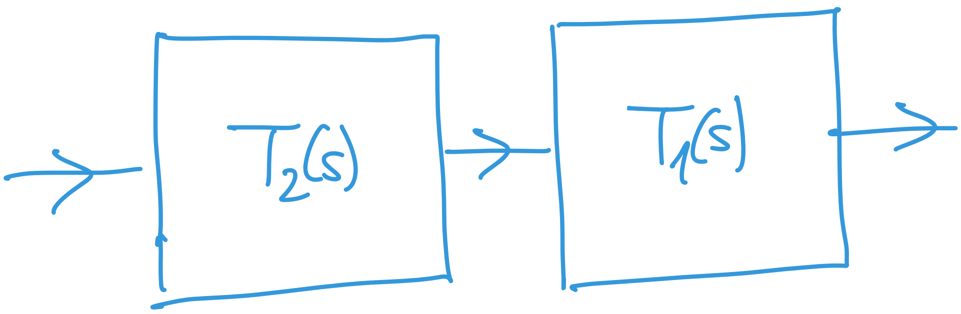


[+1 point]

Part III

No matter how we connect the circuits in part II, there is <u>loading</u>. [+1 point]



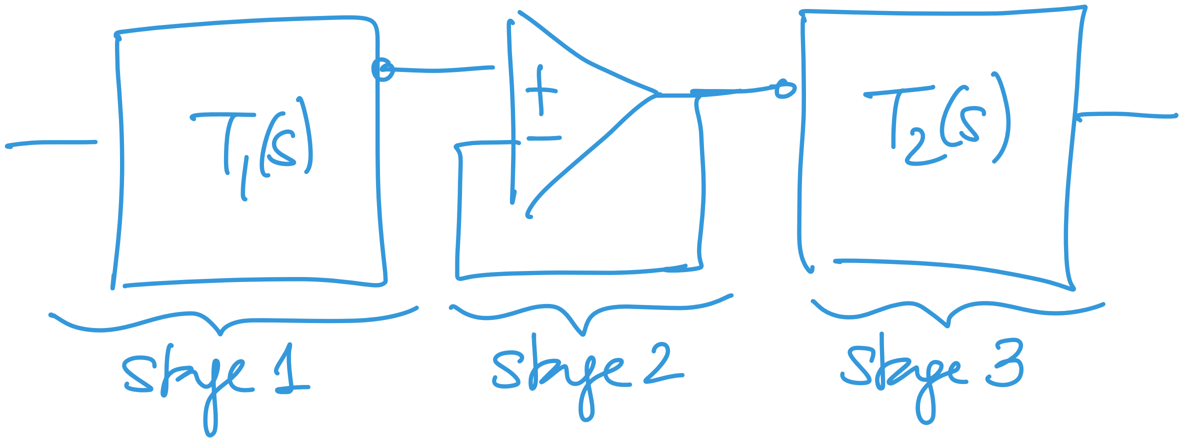


This means ~~that~~ the actual transfer function for the interconnection is <u>not</u> $T_1(s) \cdot T_2(s)$. This explains why, after filtering $v_{in}(t)$ the output is not exactly the same as the one we computed in Q3. Part IV. [+1 point]

Part IV

To avoid loading, we could use a voltage follower. The gain of a VF is 1.

[+1 point]



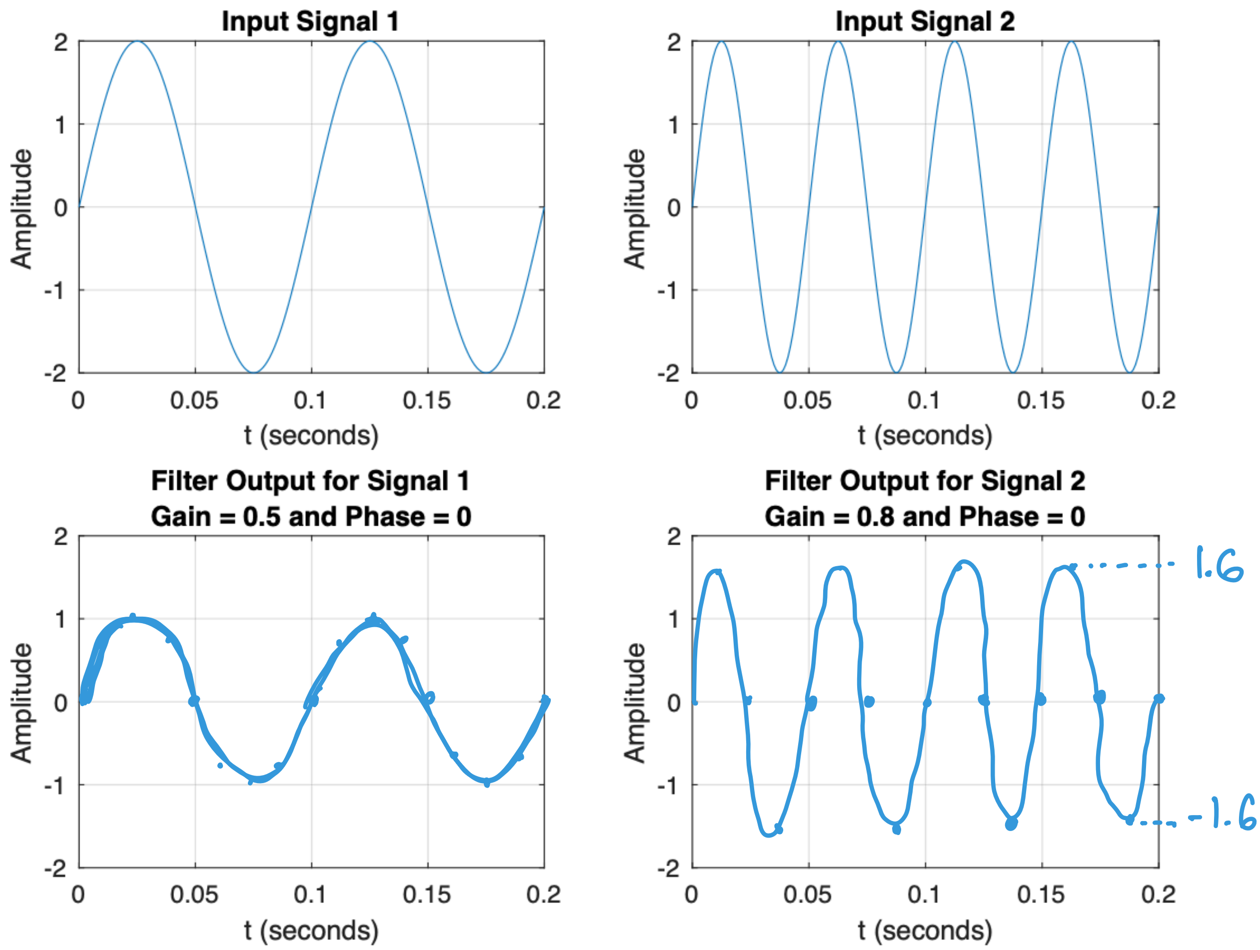
With this design, stage 2 <u>does not load</u> stage 1, because of the ∞-input impedance of the op-amp. Also, stage 3 <u>does not load</u> stage 2 because of the zero-output impedance of the op-amp. Therefore, the transfer function of the interconnection is

$$T(s) = T_1(s) \cdot 1 \cdot T_2(s) = T_1(s) \cdot T_2(s)$$

as the engineer had originally intended. This will make the output of the filtered signal equal to our answer in Q3. Part IV.

[+1 point]

5.- Part I



[+1 extra point]

Part II

It is not a low-pass filter, because the signal with smaller frequency has a significantly smaller gain than the signal with higher frequency.

[+0.5 extra point]

It could be a high-pass filter. It could also be a bandpass or a bandstop filter.

[+0.5 extra point]

6.- - Part I

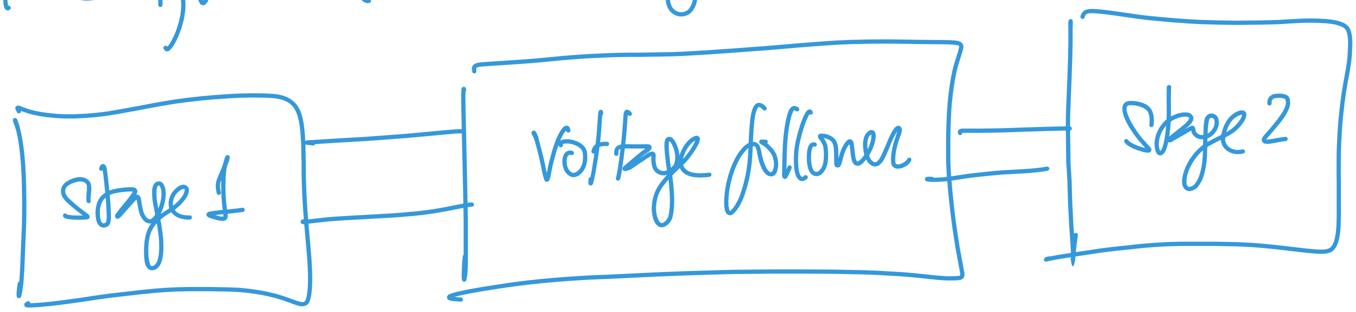
This is a voltage follower. We therefore know that

$$V_o(t) = V_i(t) \quad \text{(the gain is 1)}$$

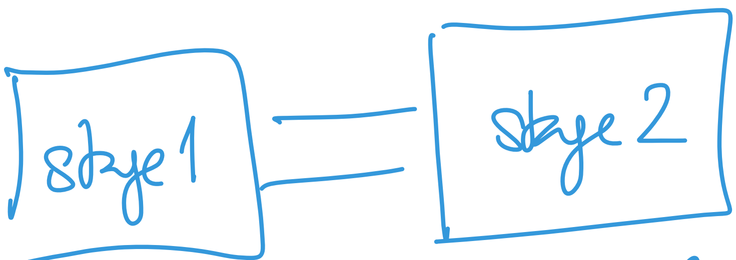
[+1 extra point]

Part II

The question asks why using



instead of using simply



The answer is <u>to avoid loading</u>. If stage 1 does not have 0 output impedance and stage 2 does not have ∞ input impedance then simply connecting the stages creates loading.

Instead, using a voltage follower in-between makes sure loading is avoided. In this way, the output voltage of stage 1 is actually delivered to stage 2. We have used this in Q4. Part IV.

[+1 extra point]