

s-Domain Circuit Analysis

Operate directly in the s -domain with capacitors, inductors and resistors

Key feature – linearity is preserved

Ccts described by ODEs and their ICs

Order equals number of C plus number of L

Element-by-element and source transformation

Nodal or mesh analysis for s -domain cct variables

Solution via Inverse Laplace Transform

Why?

1. Easier than ODEs
2. Easier to perform engineering design
3. Frequency response ideas - filtering

Element Transformations

Voltage source

Time domain

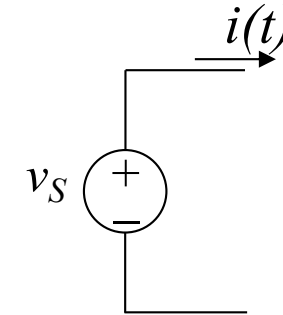
$$v(t) = v_S(t)$$

$$i(t) = \text{depends on cct}$$

Transform domain

$$V(s) = V_S(s) = \mathcal{L}(v_S(t))$$

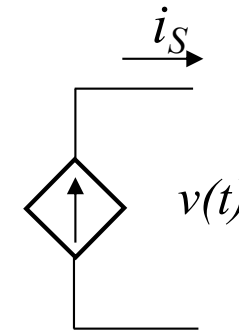
$$I(s) = \mathcal{L}(i(t)) \text{ depends on cct}$$



Current source

$$I(s) = \mathcal{L}(i_S(t))$$

$$V(s) = \mathcal{L}(v(t)) \text{ depends on cct}$$



Element Transformations contd

Controlled sources

$$v_1(t) = \mu v_2(t) \Leftrightarrow V_1(s) = \mu V_2(s)$$
$$i_1(t) = \beta i_2(t) \Leftrightarrow I_1(s) = \beta I_2(s)$$
$$v_1(t) = r i_2(t) \Leftrightarrow V_1(s) = r I_2(s)$$
$$i_1(t) = g v_2(t) \Leftrightarrow I_1(s) = g V_2(s)$$

Short cct, open cct, OpAmp relations

$$v_{SC}(t) = 0 \Leftrightarrow V_{SC}(s) = 0$$
$$i_{OC}(t) = 0 \Leftrightarrow I_{OC}(s) = 0$$
$$v_N(t) = v_P(t) \Leftrightarrow V_N(s) = V_P(s)$$

Sources and active devices behave identically

Constraints expressed between transformed variables

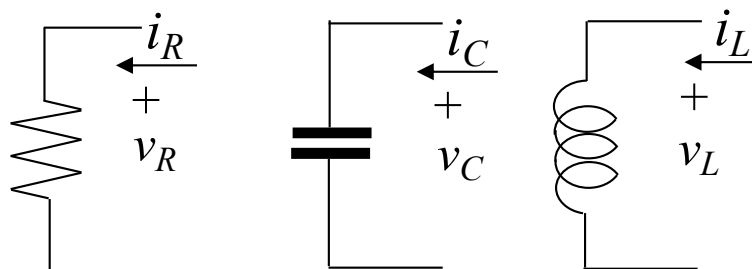
This all hinges on uniqueness of Laplace Transforms and linearity

Element Transformations contd

Resistors

$$v_R(t) = Ri_R(t) \quad i_R(t) = Gv_R(t)$$

$$V_R(s) = RI_R(s) \quad I_R(s) = GV_R(s)$$



$$Z_R(s) = R \quad Y_R(s) = \frac{1}{R}$$

Capacitors

$$i_C(t) = C \frac{dv_C(t)}{dt} \quad v_C(t) = \frac{1}{C} \int_0^t i_C(\tau) d\tau + v_C(0)$$

$$I_C(s) = sCV_C(s) - Cv_C(0^-) \quad V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s}$$

$$Z_C(s) = \frac{1}{sC} \quad Y_C(s) = sC$$

Inductors

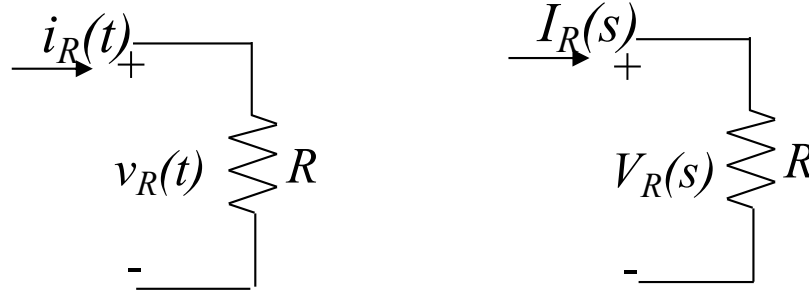
$$v_L(t) = L \frac{di_L(t)}{dt} \quad i_L(t) = \frac{1}{L} \int_0^t v_L(\tau) d\tau + i_L(0)$$

$$V_L(s) = sLI_L(s) - Li_L(0) \quad I_L(s) = \frac{1}{sL} V_L(s) + \frac{i_L(0)}{s}$$

$$Z_L(s) = sL \quad Y_L(s) = \frac{1}{sL}$$

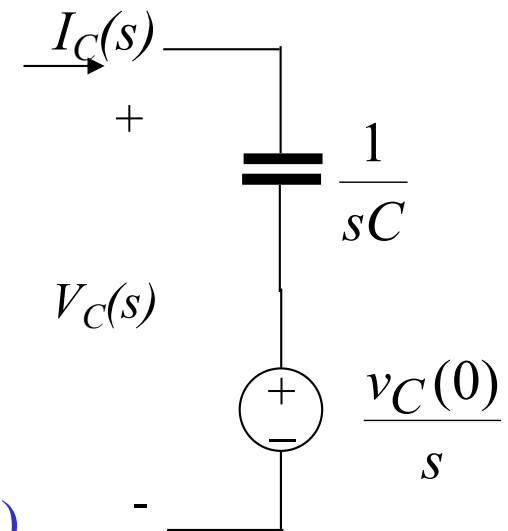
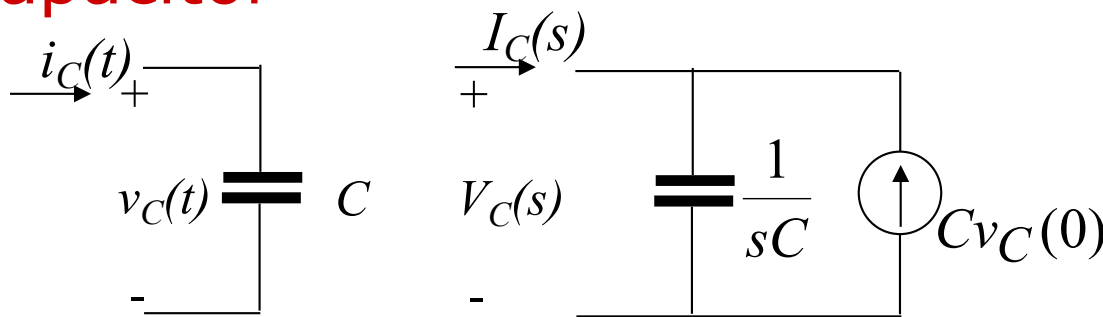
Element Transformations contd

Resistor



$$V_R(s) = RI_R(s)$$

Capacitor



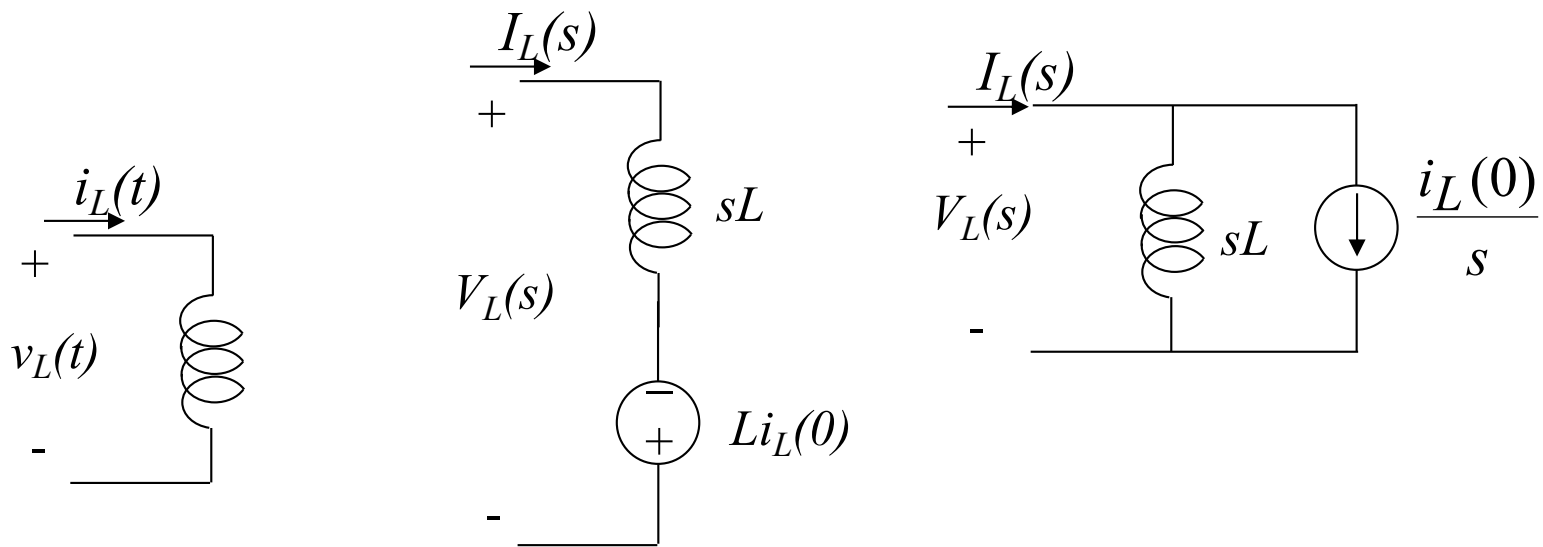
$$I_C(s) = sCV_C(s) - Cv_C(0^-) \quad V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s}$$

Note the source transformation rules apply!

Element Transformations contd

Inductors

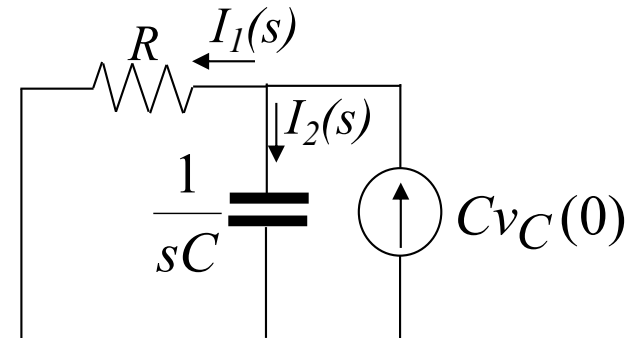
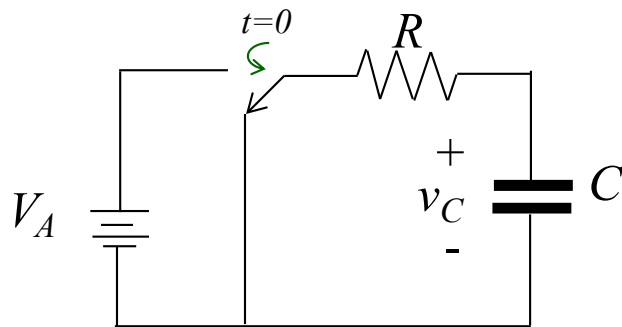
$$V_L(s) = sLI_L(s) - Li_L(0) \quad I_L(s) = \frac{1}{sL}V_L(s) + \frac{i_L(0)}{s}$$



Example 10-1, T&R, 5th ed, p 456

RC cct behavior

Switch in place since $t = -\infty$, closed at $t = 0$. Solve for $v_C(t)$.



Initial conditions $v_C(0) = V_A$

s-domain solution using nodal analysis

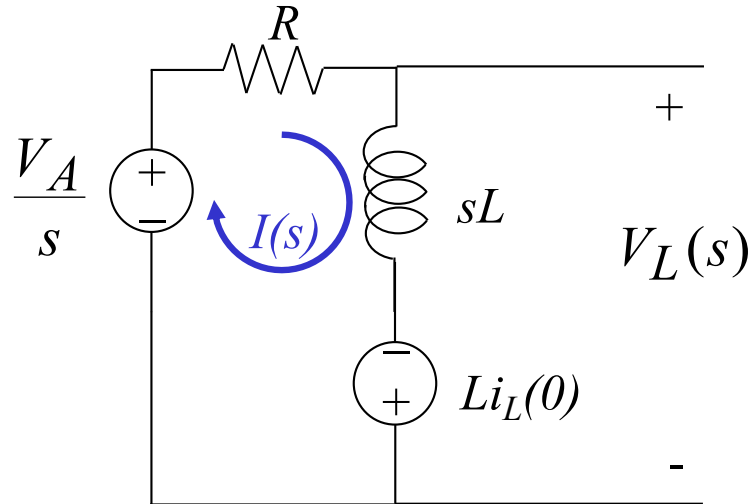
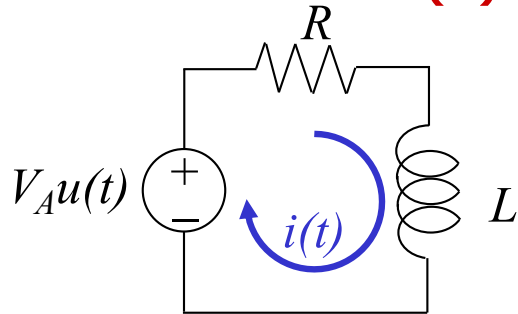
$$I_1(s) = \frac{V_C(s)}{R} \quad I_2(s) = \frac{V_C(s)}{1/sC} = sCV_C(s)$$

t-domain solution via inverse Laplace transform

$$V_C(s) = \frac{V_A}{s + \frac{1}{RC}} \quad v_C(t) = V_A e^{-t/RC} u(t)$$

Example 10-2 T&R, 5th ed, p 457

Solve for $i(t)$



KVL around loop $\frac{V_A}{s} - (R + sL)I(s) + Li_L(0) = 0$

Solve
$$I(s) = \frac{V_A/L}{s(s + R/L)} + \frac{i_L(0)}{s + R/L} = \frac{V_A/R}{s} + \frac{(i_L(0) - V_A/R)}{s + R/L}$$

Invert
$$i(t) = \left[\frac{V_A}{R} - \frac{V_A}{R} e^{-Rt/L} + i_L(0) e^{-Rt/L} \right] u(t) \text{ Amps}$$

Impedance and Admittance

Impedance (Z) is the s -domain proportionality factor relating the transform of the voltage across a two-terminal element to the transform of the current through the element with all initial conditions zero

$$V(s) = Z(s)I(s)$$

Admittance (Y) is the s -domain proportionality factor relating the transform of the current through a two-terminal element to the transform of the voltage across the element with initial conditions zero

$$I(s) = Y(s)V(s)$$

Impedance is like resistance

Admittance is like conductance

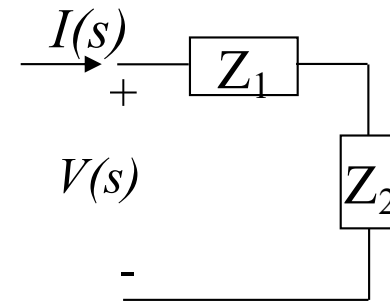
Circuit Analysis in s-Domain

Basic rules

The equivalent **impedance** $Z_{eq}(s)$ of two impedances $Z_1(s)$ and $Z_2(s)$ in series is $Z_{eq}(s) = Z_1(s) + Z_2(s)$

Same current flows

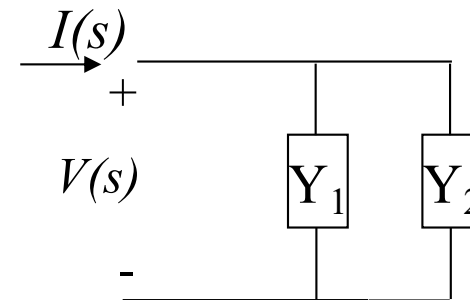
$$V(s) = Z_1(s)I(s) + Z_2(s)I(s) = Z_{eq}(s)I(s)$$



The equivalent **admittance** $Y_{eq}(s)$ of two admittances $Y_1(s)$ and $Y_2(s)$ in parallel is $Y_{eq}(s) = Y_1(s) + Y_2(s)$

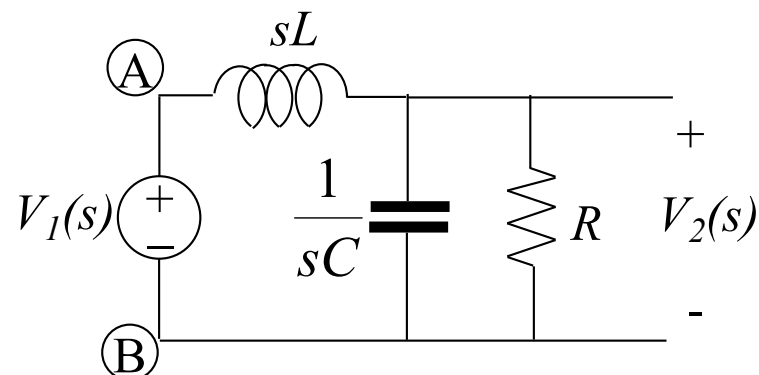
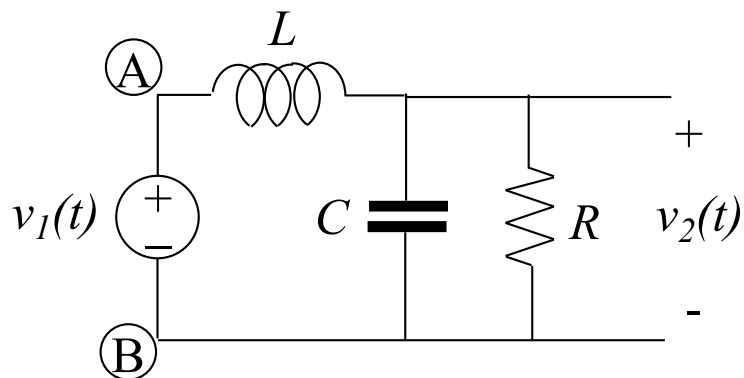
Same voltage

$$I(s) = Y_1(s)V(s) + Y_2(s)V(s) = Y_{eq}(s)V(s)$$



Example 10-3 T&R, 5th ed, p 461

Find $Z_{AB}(s)$ and then find $V_2(s)$ by voltage division
 ($v_C(0)=0$ and $i_L(0)=0$)

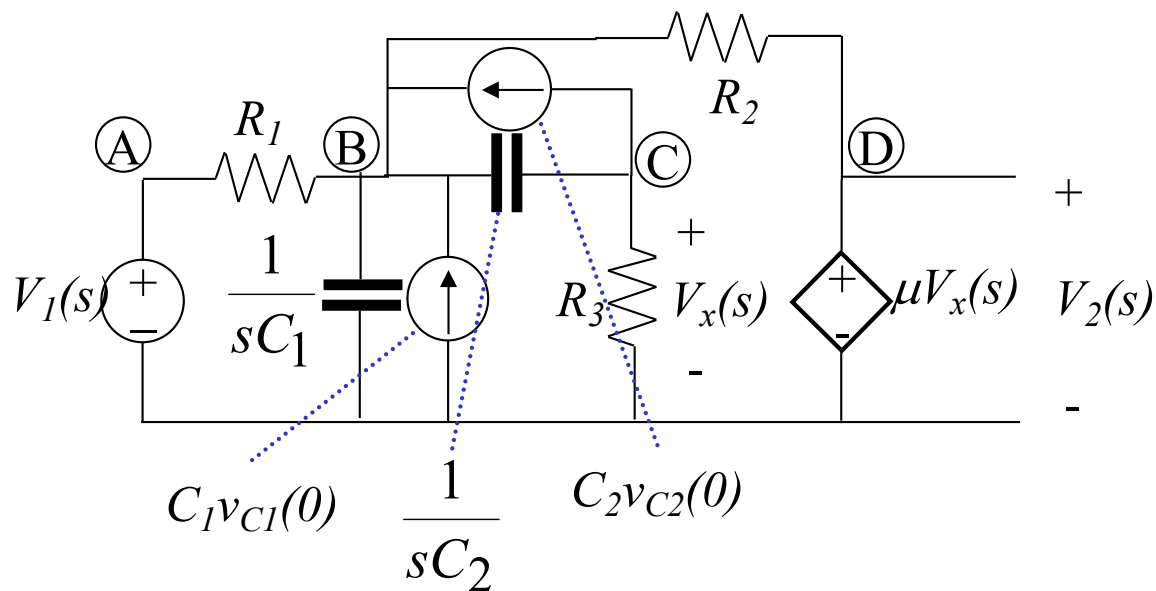
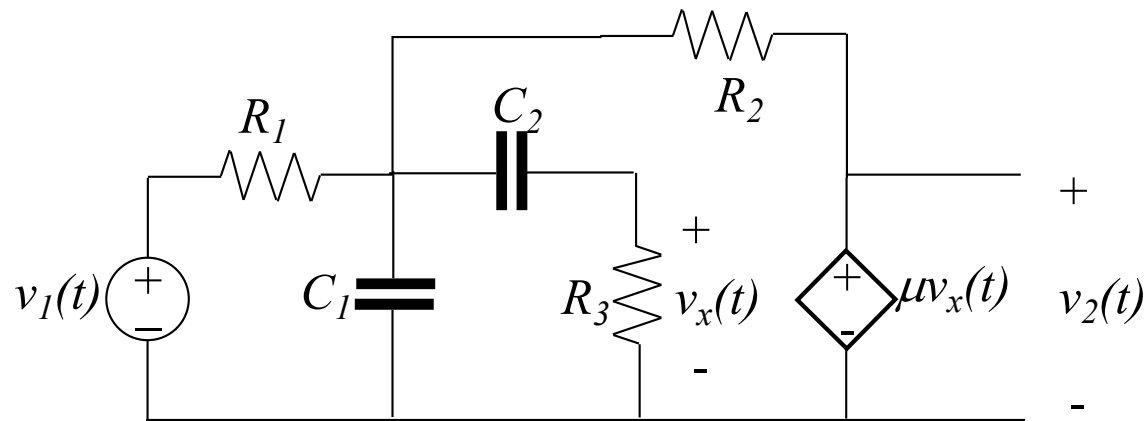


$$Z_{eq}(s) = sL + R \parallel \frac{1}{sC} = sL + \frac{1}{\frac{1}{R} + sC} = \frac{RLCs^2 + Ls + R}{RCs + 1} \Omega$$

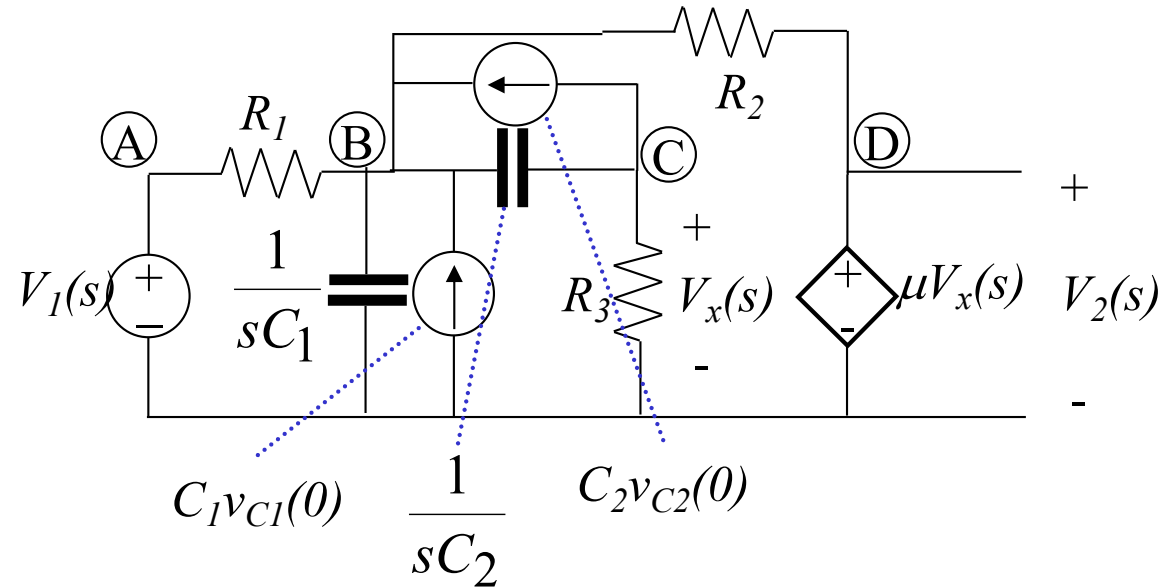
$$V_2(s) = \left[\frac{Z_1(s)}{Z_{eq}(s)} \right] V_1(s) = \left[\frac{R}{RLCs^2 + sL + R} \right] V_1(s)$$

Example

Formulate node voltage equations in the s-domain



Example contd



Node A: $V_A(s) = V_1(s)$

Node D: $V_D(s) = \mu V_x(s) = \mu V_C(s)$

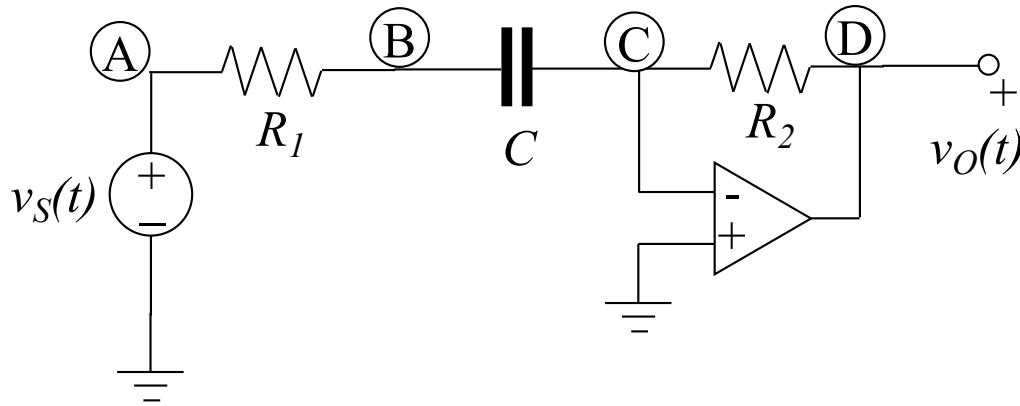
Node B:

$$\frac{V_B(s) - V_A(s)}{R_1} + \frac{V_B(s) - V_D(s)}{R_2} + \frac{V_B(s)}{1/sC_1} + \frac{V_B(s) - V_C(s)}{1/sC_2} - C_1 v_{C1}(0) - C_2 v_{C2}(0) = 0$$

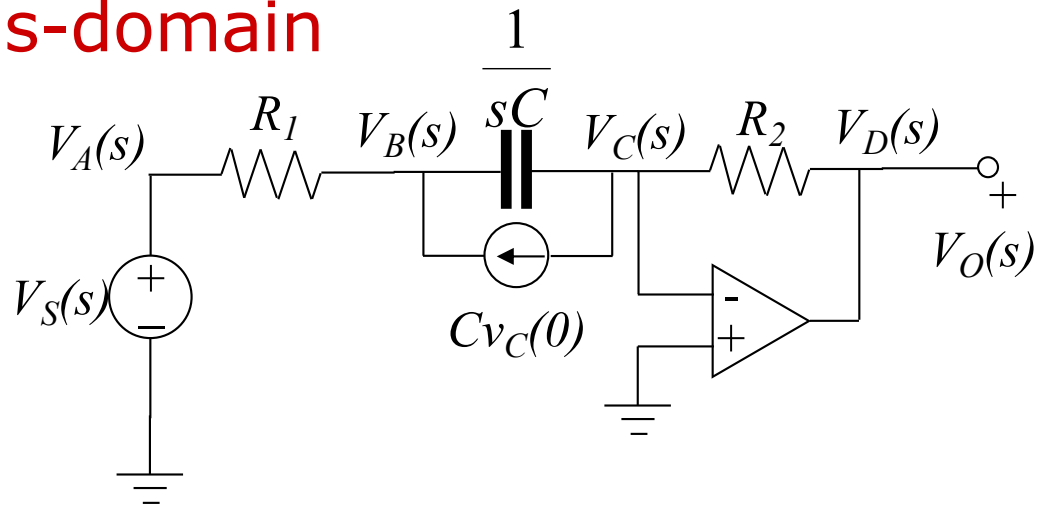
Node C: $-sC_2 V_B(s) + [sC_2 + G_3] V_C(s) = -C_2 v_{C2}(0)$

Example

Find $v_O(t)$ when $v_S(t)$ is a unit step $u(t)$ and $v_C(0)=0$



Convert to s-domain



Example

Nodal Analysis

Node A: $V_A(s) = V_S(s)$

Node D: $V_D(s) = V_O(s)$

Node C: $V_C(s) = 0$

Node B: $(G_1 + sC)V_B(s) - G_1V_S(s) = Cv_C(0)$

Node C KCL: $-sCV_B(s) - G_2V_O(s) = -Cv_C(0)$

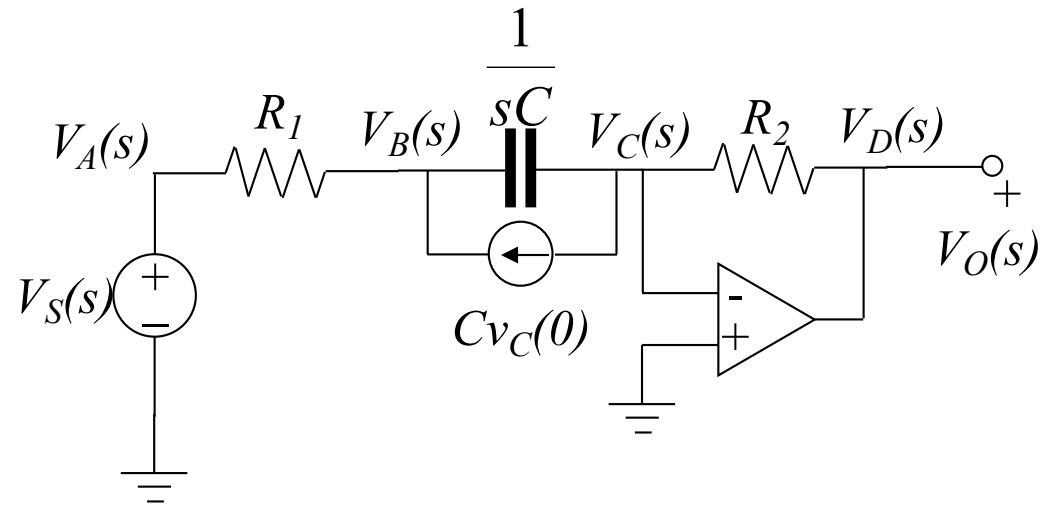
Solve for $V_O(s)$

$$V_O(s) = - \left[\frac{sG_1C / G_2}{G_1 + sC} \right] V_S(s) = - \left[\frac{R_2}{R_1} \times \frac{s}{s + 1/R_1C} \right] V_S(s)$$

$$= - \left[\frac{R_2}{R_1} \times \frac{s}{s + 1/R_1C} \right] \frac{1}{s} = \frac{-R_2}{R_1} \times \frac{1}{s + 1/R_1C}$$

Invert LT

$$v_O(t) = \frac{-R_2}{R_1} e^{-\frac{t}{R_1C}} u(t)$$



Superposition in s-domain ccts

The s-domain response of a cct can be found as the sum of two responses

1. The zero-input response caused by initial condition sources, with all external **inputs** turned **off**
2. The zero-state response caused by the external sources, with **initial condition** sources set to **zero**

Linearity and superposition

Another subdivision of responses

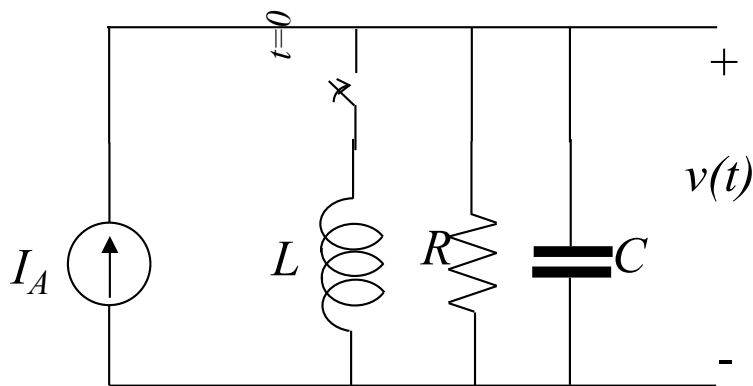
1. Natural response – the general solution
Response representing the natural modes (poles) of cct
2. Forced response – the particular solution
Response containing modes due to the input

Example 10-6, T&R, 5th ed, p 466

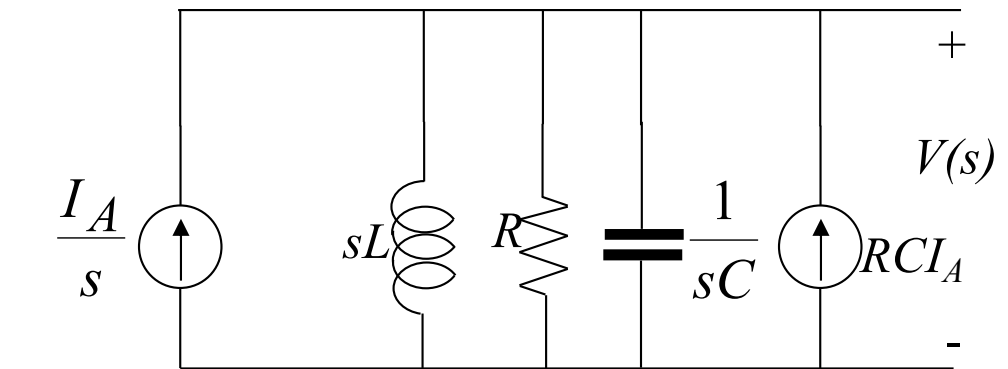
The switch has been open for a long time and is closed at $t=0$.

Find the zero-state and zero-input components of $V(s)$

Find $v(t)$ for $I_A=1\text{mA}$, $L=2\text{H}$, $R=1.5\text{K}\Omega$, $C=1/6\ \mu\text{F}$



$$Z_{eq}(s) = \frac{1}{\frac{1}{sL} + \frac{1}{R} + sC} = \frac{RLs}{RLCs^2 + Ls + R}$$



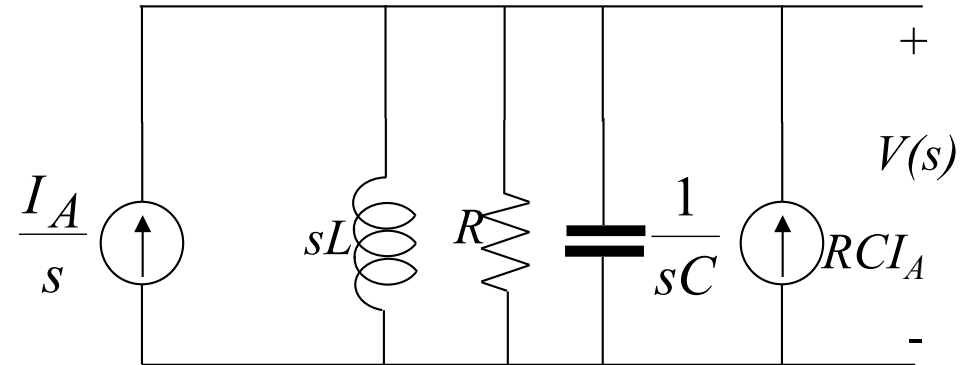
$$V_{zs}(s) = Z_{eq}(s) \frac{I_A}{s} = \frac{I_A/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V_{zi}(s) = Z_{eq}(s) RCI_A = \frac{RI_A s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Example 10-6 contd

$$V_{zs}(s) = Z_{eq}(s) \frac{I_A}{s} = \frac{I_A / C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V_{zi}(s) = Z_{eq}(s) RCI_A = \frac{RI_A s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$



Substitute values

$$V_{zs}(s) = \frac{6000}{(s+1000)(s+3000)} = \frac{3}{s+1000} + \frac{-3}{s+3000}$$

$$v_{zs}(t) = \left[3e^{-1000t} - 3e^{-3000t} \right] u(t)$$

$$V_{zi}(s) = \frac{1.5s}{(s+1000)(s+3000)} = \frac{-0.75}{s+1000} + \frac{2.25}{s+3000}$$

$$v_{zi}(t) = \left[-0.75e^{-1000t} + 2.25e^{-3000t} \right] u(t)$$

What are the natural and forced responses?

Features of s-domain cct analysis

The response transform of a finite-dimensional, lumped-parameter linear cct with input being a sum of exponentials is a rational function and its inverse Laplace Transform is a sum of exponentials

The exponential modes are given by the poles of the response transform

Because the response is real, the poles are either real or occur in complex conjugate pairs

The natural modes are the zeros of the cct determinant and lead to the natural response

The forced poles are the poles of the input transform and lead to the forced response

Features of s-domain cct analysis

A cct is **stable** if all of its poles are located in the open left half of the complex s-plane

A key property of a system

Stability: the natural response dies away as $t \rightarrow \infty$

Bounded inputs yield bounded outputs

A cct composed of Rs, Cs and Ls will be at worst marginally stable

With Rs in the right place it will be stable

$Z(s)$ and $Y(s)$ both have no poles in $\text{Re}(s) > 0$

Impedances/admittances of RLC ccts are “Positive Real” or energy dissipating