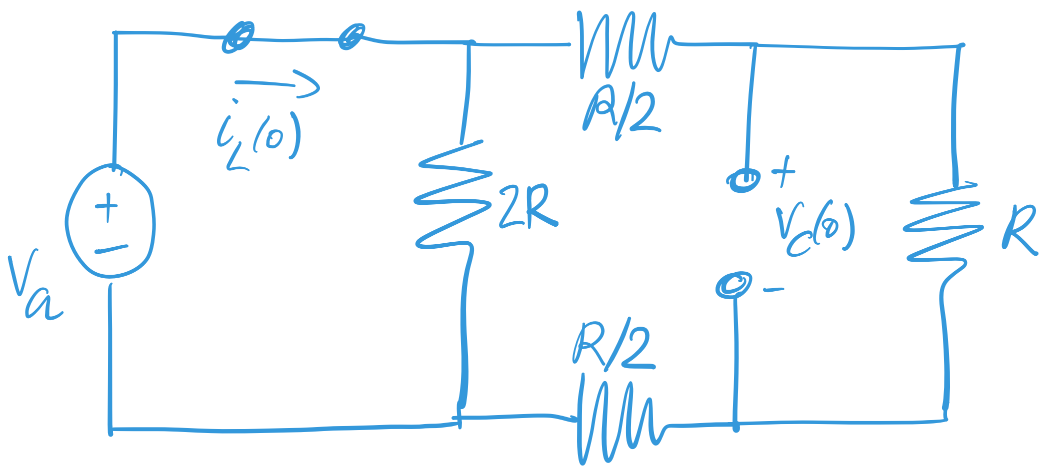
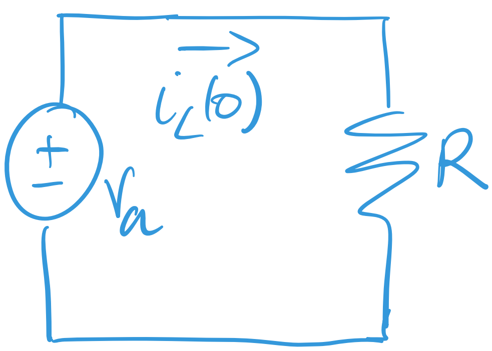
1._ Part I

Under DC excitations, we know the capacitor behaves as an open circuit and the inductor behaves as a short circuit. Therefore we have



[+1 point for correct drawing]

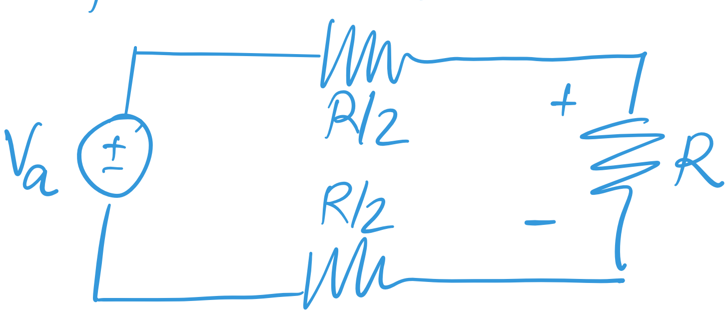
To find out $i_L(0)$, we combine the three resistors in series. The resulting resistor is in parallel with 2R.



Therefore, $i_L(0) = \frac{V_a}{R}$

[+1.5 point]

To find out $V_C(0)$, we draw the equivalent circuit



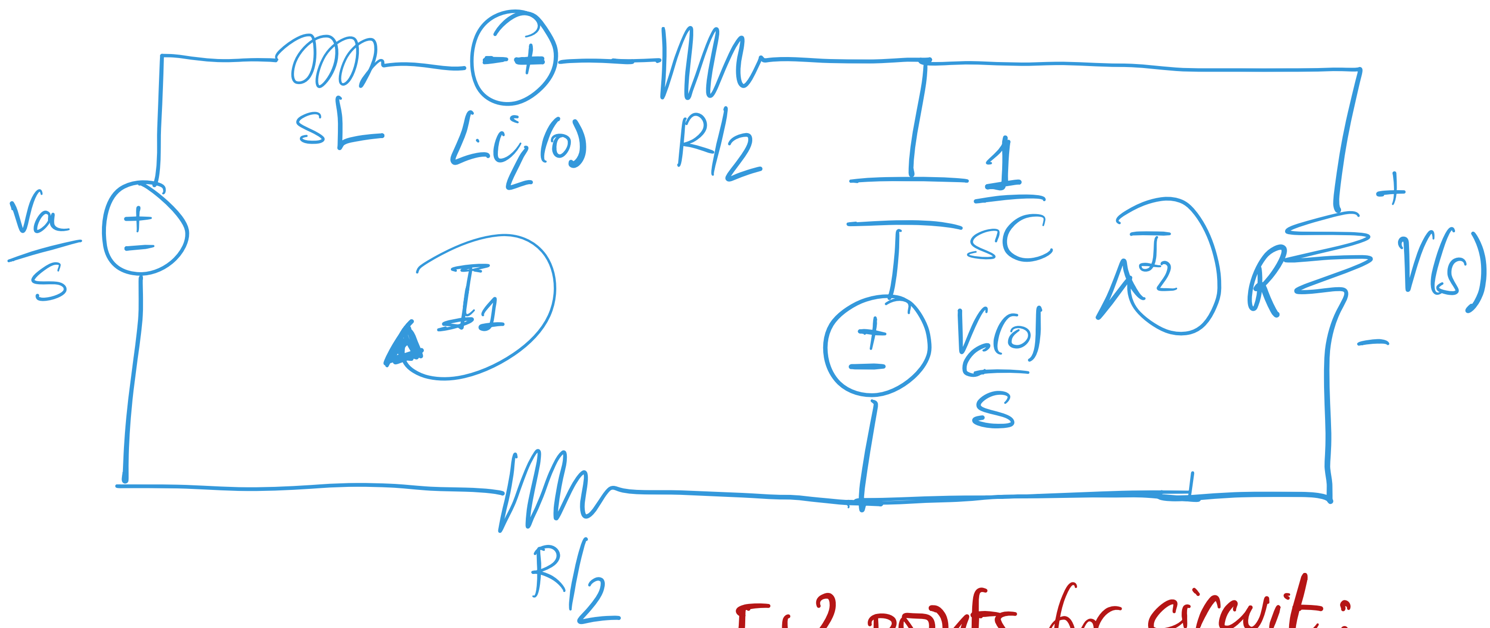
By voltage division,

$$V_C(0) = \frac{R}{2R} V_a = \frac{V_a}{2}$$

[+1.5 point]

Part II

We redraw the circuit in the s-domain, using voltage sources to account for the initial conditions of the inductor and the capacitor.



[+2 points for circuit; +2 point for correctly capturing initial conditions]

Part III

We use mesh current analysis, as instructed. We write equations by inspection

$$\begin{pmatrix} sL+R+\frac{1}{sC} & -1/sC \\ -1/sC & \frac{1}{sC}+R \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \frac{V_a}{s}+Li_L(0)-\frac{V_C(0)}{s} \\ \frac{V_C(0)}{s} \end{pmatrix}$$

[+2 points]

Part IV

We simply have

$$V(s) = R \cdot I_2(s)$$ [+0.5 bonus point]

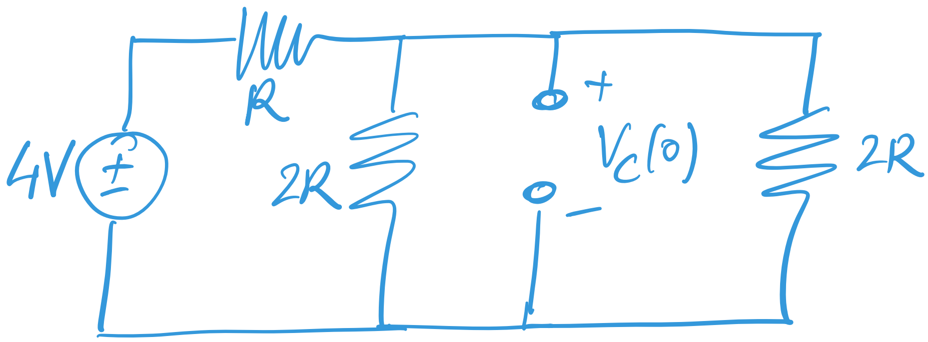
This is the same as the transform of the capacitor voltage $V_C(s)$. Therefore

$$V(s) - \frac{V_C(0)}{s} = V_C(s) - \frac{V_C(0)}{s}$$

is the voltage drop across the impedance $\frac{1}{Cs}$. [+0,5 bonus point]

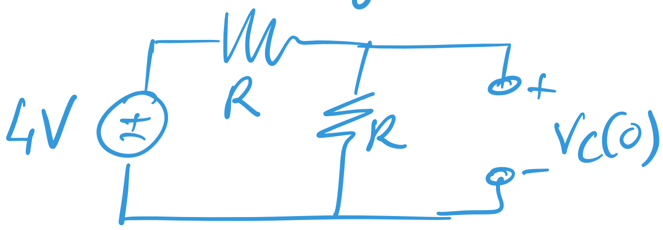
2.- Part I

To find the initial condition, we substitute the capacitor by an open circuit



[+1 point]

Combining the resistors in parallel,



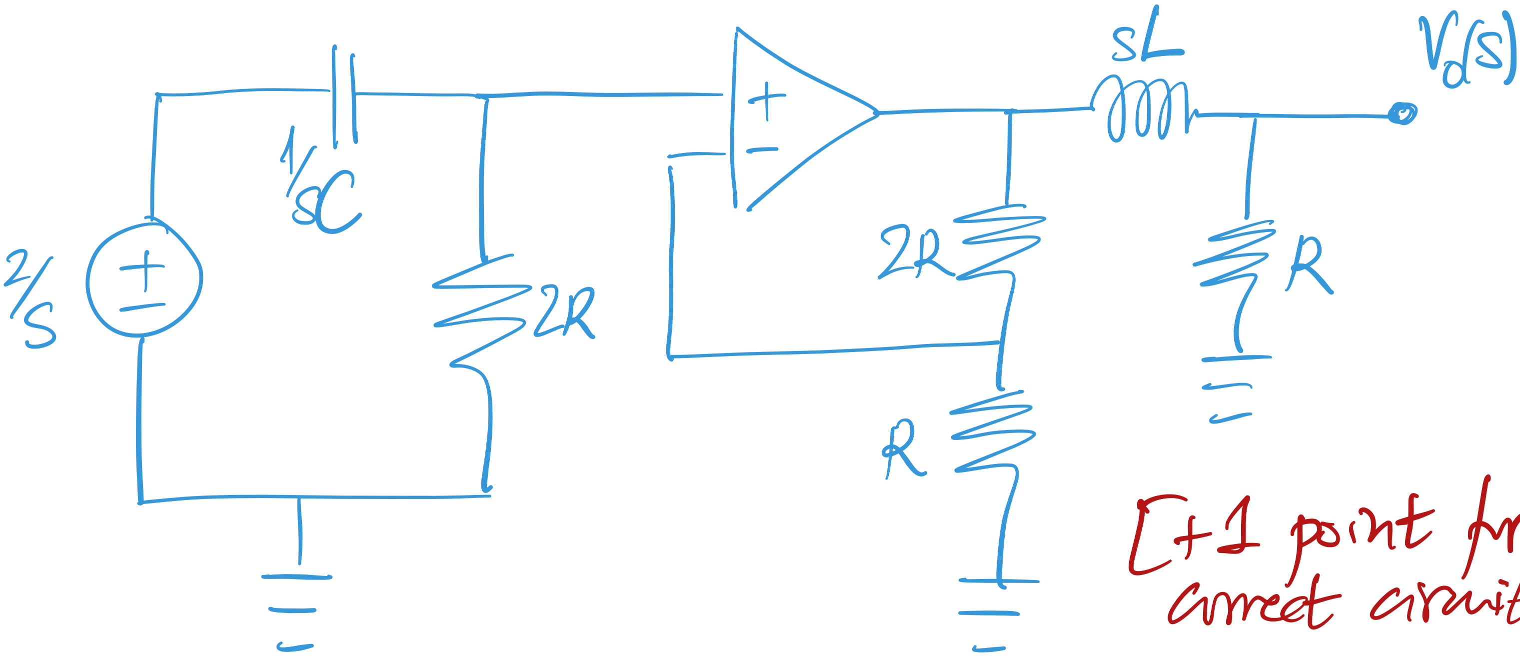
Therefore, using voltage division,

$$V_c(0) = \frac{R}{R+R} 4V = 2V$$

[+1 point]

Part II

We add a voltage source, as instructed, to take care of the initial condition of the capacitor. We need to carefully take into account the polarities to get the transformation right. Also, the initial condition for the inductor is zero, so no need to worry about it.



[+1 point for correct circuit]

The transfer function can be found by realizing this is the composition of a voltage divider, a non-inverting op-amp, and another voltage divider. The presence of the non-inverting op-amp ensures there is no loading.

[+1 point]

Therefore,

$$T(s) = \frac{2R}{2R + 1/sC} \cdot \frac{2R+R}{R} \cdot \frac{R}{R+sL} =$$

$$= \frac{6R^2Cs}{(2RCs+1)(R+sL)}$$

[+1 point]

Part III

Using the expression for the transfer function w/ the input $\frac{2}{s}$, we have

$$V_0(s) = T(s) \cdot \frac{2}{s} = \frac{12R^2C}{(2RCs+1)(R+sL)}$$ [+1 point]

We substitute the values provided to obtain

$$V_0(s) = \frac{18}{(s+1)(3+s)}$$

To find the inverse Laplace transform, we use partial fraction decomposition

$$V_0(s) = \frac{A}{s+1} + \frac{B}{s+3}$$ [+1 point]

We use the residue method,

$$A = \lim_{s \to -1} (s+1)V_0(s) = \lim_{s \to -1} \frac{18}{s+3} = 9$$

$$B = \lim_{s \to -3} (s+3)V_o(s) = \lim_{s \to -3} \frac{18}{s+1} = -9$$

Therefore, $V_o(s) = \dfrac{9}{s+1} - \dfrac{9}{s+3}$

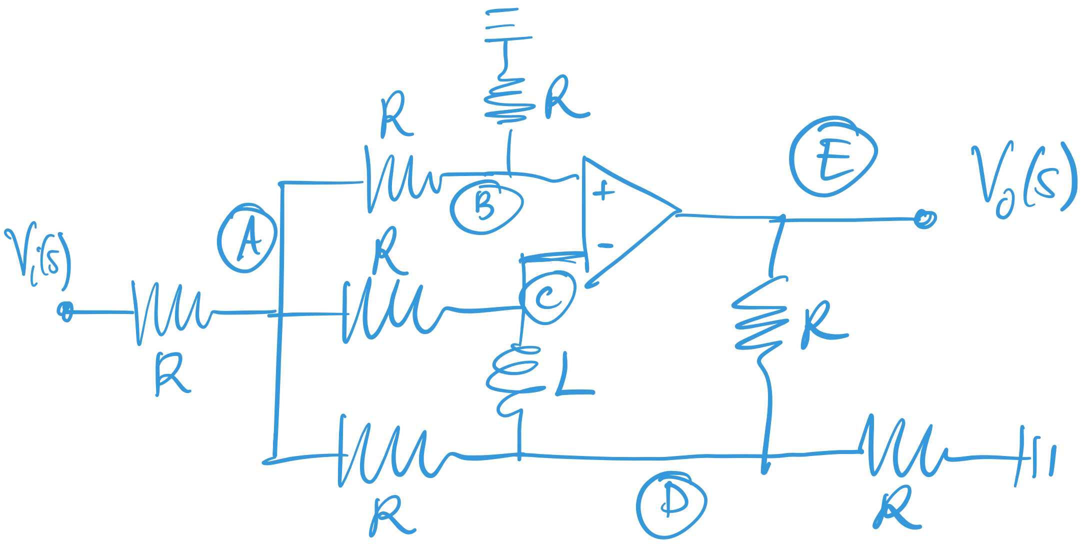
[+2 points]

The output voltage is then

$$v_o(t) = \left(9e^{-t} - 9e^{-3t}\right) u(t)$$

[+1 point]

3.- **Part I** Since there are no initial conditions, no need to add independent source b/c of the inductor.



[+1 point]

Part II

We use nodal analysis.

We know we should not write KCL for the output node of the op-amp. Therefore, we write KCL for nodes A through D, and use ideal op-amp conditions.

KCL @ A

$$\frac{1}{R}\left(V_A(s)-V_i(s)\right)+\frac{1}{R}\left(V_A(s)-V_B(s)\right)+\frac{1}{R}\left(V_A(s)-V_C(s)\right)+$$

$$\frac{1}{R}\left(V_A(s)-V_D(s)\right)=0$$

[+0.5 point]

KCL @ B

$$\frac{1}{R}\left(V_B(s)-V_A(s)\right)+\frac{1}{R}V_B(s)=0$$ [+0,5 point]

KCL @ C

$$\frac{1}{R}\left(V_C(s)-V_A(s)\right)+\frac{1}{sL}\left(V_C(s)-V_D(s)\right)=0$$

[+0,5 point]

KCL @ D

$$\frac{1}{sL}\left(V_D(s)-V_C(s)\right)+\frac{1}{R}\left(V_D(s)-V_A(s)\right)+\frac{1}{R}V_D(s)$$

$$+\frac{1}{R}\left(V_D(s)-V_E(s)\right)=0$$ [+0,5 point]

Additionally, ideal op-amp conditions give

[+0,5 point]

$$V_B(s)=V_C(s)$$

Finally, note that $V_o(s)=V_E(s)$ [+0,5 point]

Solving the above system of equations, we get

$$V_o(s)=\boxed{-\frac{3Ls}{5R+Ls}}\,V_i(s)$$

$= T(s)$

Part III With the provided values, the transfer function takes the form

$$T(s) = \frac{-3\cdot 10^{-2} s}{5\cdot 10^{-1} + 10^{-2} s} = -\frac{3s}{50+s}$$

We evaluate at $s = j\omega$,

$$T(j\omega) = \frac{-3j\omega}{50 + j\omega}, \quad \omega \geq 0$$

Its magnitude is the gain function

$$|T(j\omega)| = \frac{3\omega}{\sqrt{2500 + \omega^2}}$$

[+0,5 point]

And its phase

$$\angle T(j\omega) = \angle(-3j\omega) - \angle(50 + j\omega) =$$

$$= \frac{3\pi}{2} - \arctan\left(\frac{\omega}{50}\right)$$

[+0,5 point]

At $t=0$,

$|T(j0)| = 0$ $\quad \angle T(j0) = \frac{3\pi}{2} - 0 = \frac{3\pi}{2}$

[+0,5 point]

At $t=\infty$

$|T(j\infty)| = 3$ $\quad \angle T(j\infty) = \frac{3\pi}{2} - \frac{\pi}{2} = \pi$

[+0,5 point]

The cut-off frequency is defined by

$$|T(j\omega_c)| = \frac{T_{max}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\|$$

$$\frac{3\omega_c}{\sqrt{2500+\omega_c^2}} \Rightarrow \frac{\omega_c^2}{2500+\omega_c^2} = \frac{1}{2}$$

$$2\omega_c^2 = 2500 + \omega_c^2$$

$$\omega_c^2 = 2500$$

[+0,5 point]

$$\omega_c = 50 \text{ rad/s}$$

Finally, to sketch the magnitude of the freq. response, we consider

$\omega \ll 1$, $|T(j\omega)| \sim \frac{3\omega}{\sqrt{2500}} = \frac{3}{50}\omega$

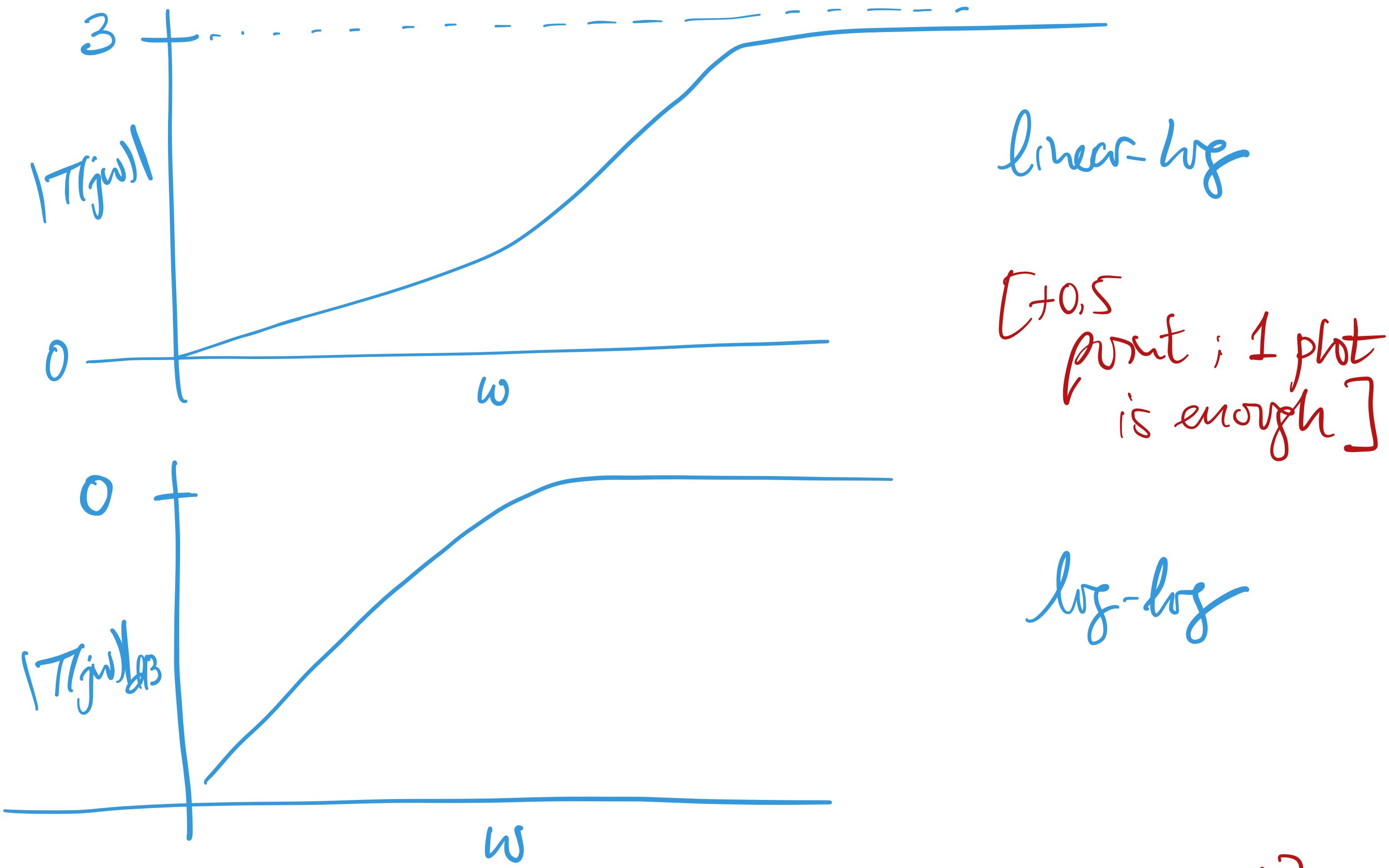
$\log \frac{3}{50}\omega$

$\|$

$\log \frac{3}{50} + \log \omega$

$$\omega \gg 1 \quad |T(j\omega)| \sim \frac{3\omega}{\sqrt{\omega^2}} = 3$$

The sketch of the gain is



This is a high-pass filter. [+1 point]

Part IV

Note that $|T(j50)| = \frac{3}{\sqrt{2}}$

$$\angle T(j50) = \frac{3\pi}{2} - \underbrace{\arctan 1}_{\frac{\pi}{4}} = \frac{5\pi}{4}$$

Therefore, the steady-state response to the input $v_i(t) = -2\cos(50t - \frac{\pi}{4})$ is

$$v_o^{ss}(t) = -2 \cdot |T(j50)| \cos\left(50t - \frac{\pi}{4} + \angle T(j50)\right) =$$

[+1 point for correct expression]

$$= -2\frac{3}{\sqrt{2}} \cos\left(50t - \frac{\pi}{4} + \frac{5\pi}{4}\right)$$

$$= -3\sqrt{2} \cos(50t + \pi) =$$

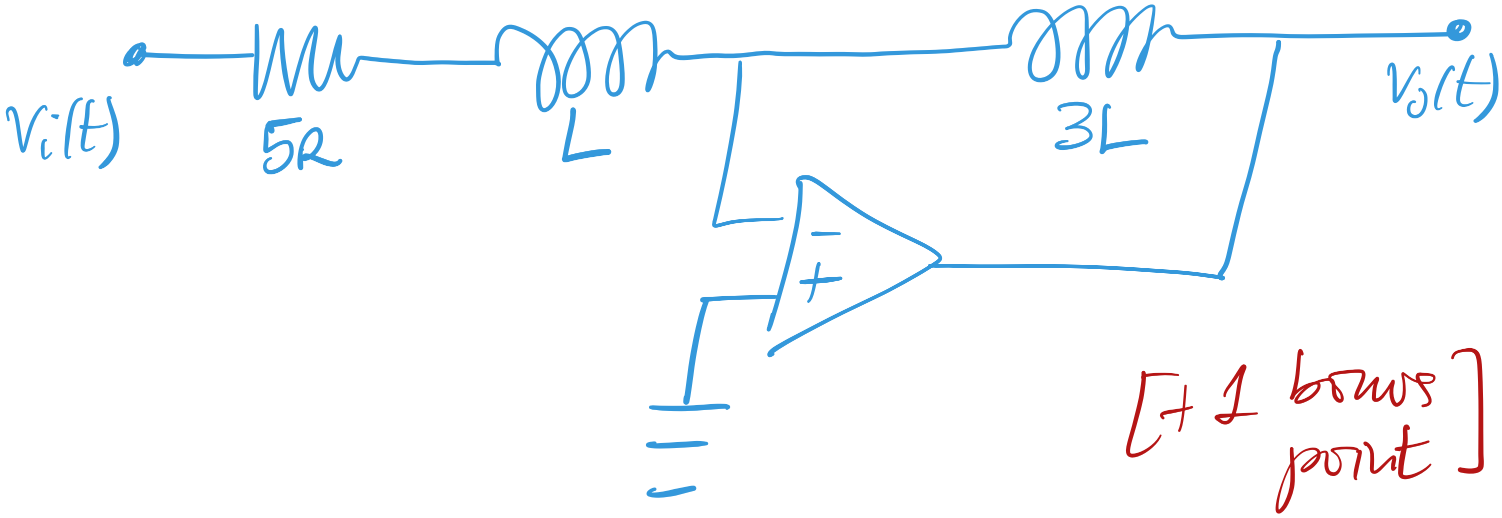
$$= 3\sqrt{2}\cos(50t)$$

[+1 point]

Part V

We can easily generate $T(s) = -\frac{3Ls}{5R + Ls}$ with an inverting op-amp, $T(s) = \frac{-Z_2(s)}{Z_1(s)}$.

We use 1 inductor of value $3L$ for $Z_2(s)$ and 1 resistor of $5R$ in series with one inductor of value L for $Z_1(s)$.



[+1 bonus point]

Both designs have 0 output impedance but finite input impedance. The inverting op-amp design seems preferable because of its simplicity compared ~~to~~ to the one in Fig. 3 and fewer number of components.

[+1 bonus point]

4.- **Part I**

The first voltage divider has transfer function

$$T_1(s) = \frac{100}{100 + \frac{1}{10^{-4}s}} = \frac{10^2 \cdot 10^{-2} s}{10^2 \cdot 10^{-2} s + \frac{10^{-2}s}{10^{-4}s}} = \frac{s}{s+100}$$

[+1 point]

The second voltage divider has transfer function

$$T_2(s) = \frac{100}{100 + 10^{-1}s} = \frac{1000}{s + 1000}$$

[+1 point]

Indeed, we have

$$T(s) = T_1(s) \cdot T_2(s)$$

Part II

The instructor was expecting that the transfer function of the interconnection in series was $T(s)$. If this were the case, then

$$v_0^{ss}(t) = |T(j500)| \cdot \cos(500t + \angle T(j500))$$

[+1 point]

We compute

$$T(j\omega) = \frac{j\omega}{j\omega + 100} \cdot \frac{1000}{j\omega + 1000}$$

$$|T(j\omega)| = \frac{\omega}{\sqrt{10^4 + \omega^2}} \cdot \frac{1000}{\sqrt{10^6 + \omega^2}}$$

$$\angle T(j\omega) = \frac{\pi}{2} - \arctan\frac{\omega}{100} + 0 - \arctan\frac{\omega}{1000}$$

$$|T(j500)| \simeq 0.877 \qquad \angle T(j\omega) \simeq -0.2662$$

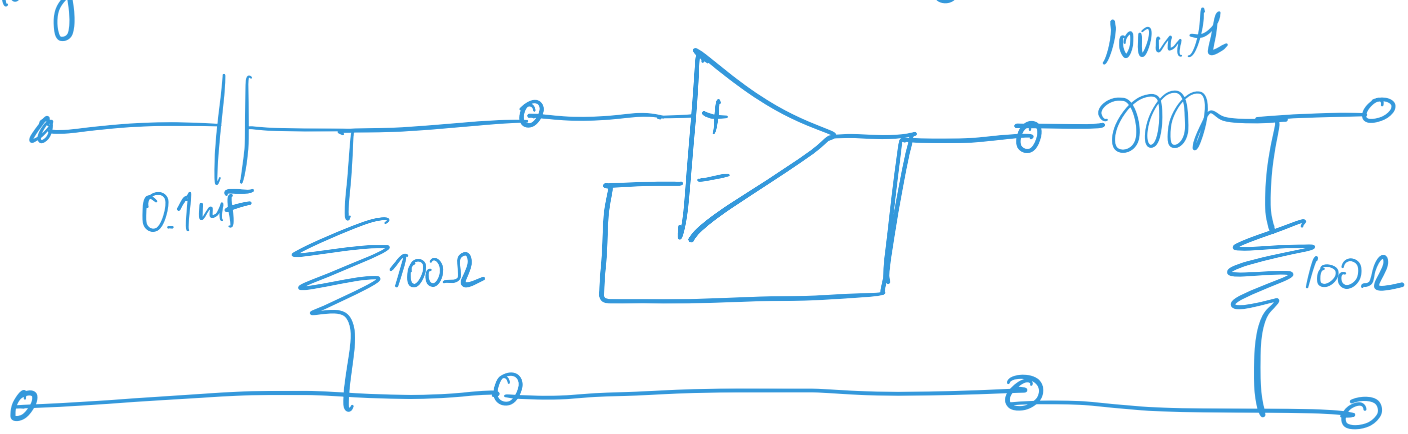
[+1 point]

That explains the steady-state response the instructor was expecting.
The reason why they did not get it is loading. No matter in which order you interconnect in series the voltage dividers, the second one always loads the first one.

[+1 point]

Part III

Yes! We could add a voltage follower (as we have discussed in class) in between the voltage dividers to avoid loading,



$$T(s) = T_1(s) \cdot 1 \cdot T_2(s) = T_1(s) \cdot T_2(s)$$

[b/c of the ∞-input impedance and 0-output impedance of voltage follower] [+1 point]

Part IV

Both designs in Figure 4(b) are inverting op-amps. [+1 point]

Therefore

$$T_1(s) = \frac{-100}{100 + 10^{-1}s} = -\frac{1000}{s+1000}$$ [+1 point]

$$T_2(s) = \frac{-100}{100 + \frac{1}{10^{-4} s}} = -\frac{s}{s+100}$$ [+1 point]

Their product is equal to $T(s)$ again.

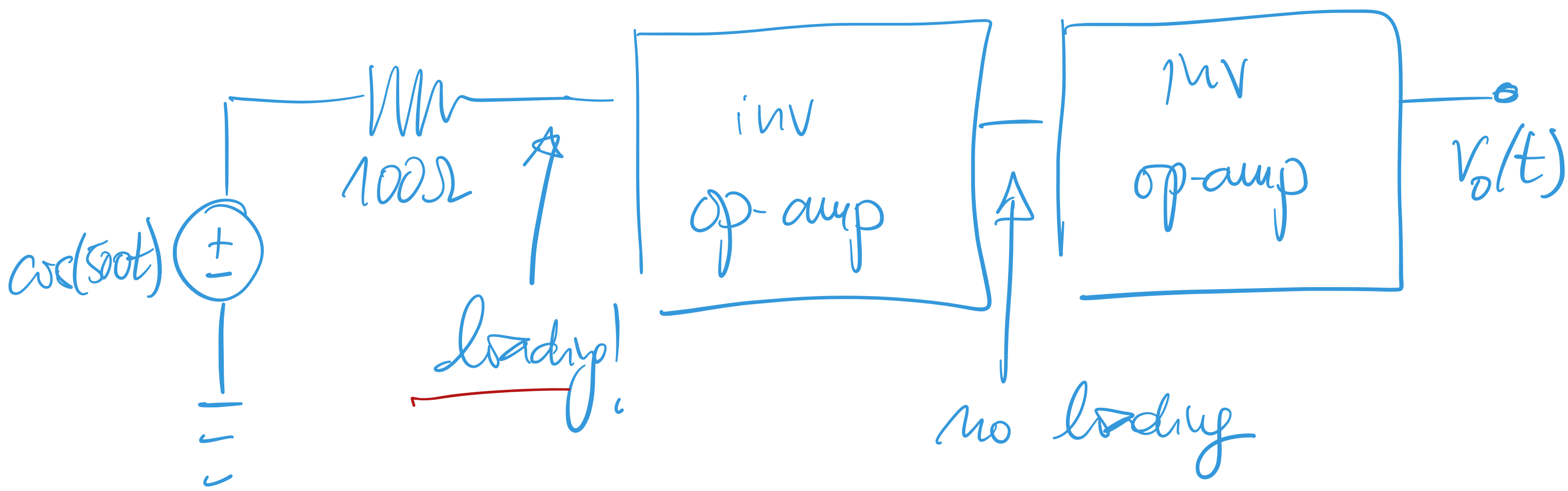
Part V

Yes! This is because in this case there is no loading, no matter in which order we connect the inverting op-amps. Both of them have 0 output impedance, so the second stage does not load the first stage. Therefore, the transfer function of the interconnection is $T(s)$, and the instructor would get the expected steady-state response. [+1 point]

Part VI

bonus [+1 point]

No! Because inverting op-amps do not have ∞-input impedance. The circuit would look like



The presence of $R_7 = 100\Omega$ together with the fact that current flows in the inv-op-amp would change the expected the steady-state output.

bonus [+1 point]