Systematic Circuit Analysis (T&R Chap 3)

Node-voltage analysis

Using the voltages of the each node relative to a ground node, write down a set of consistent linear equations for these voltages

Solve this set of equations using, say, Cramer's Rule

Mesh current analysis

Using the loop currents in the circuit, write down a set of consistent linear equations for these variables. Solve.

This introduces us to procedures for *systematically* describing circuit variables and solving for them
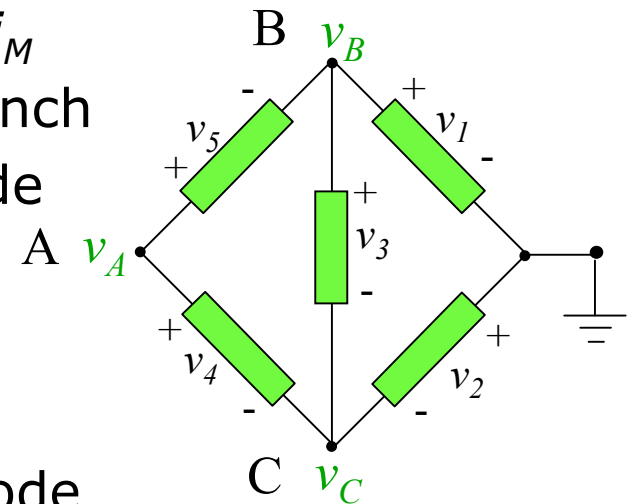
Nodal Analysis

Node voltages

Pick one node as the ground node

Label all other nodes and assign voltages v_A, v_B, ..., v_N
   and currents with each branch i_1, ..., i_M

Recognize that the voltage across a branch
   is the difference between the end node
   voltages

   Thus $v_3 = v_B - v_C$ with the direction
   as indicated

Write down the KCL relations at each node

Write down the branch *i-v* relations to express branch
   currents in terms of node voltages

Accommodate current sources

Obtain a set of linear equations for the node voltages

Nodal Analysis – Example 3-1 p.72

Apply KCL

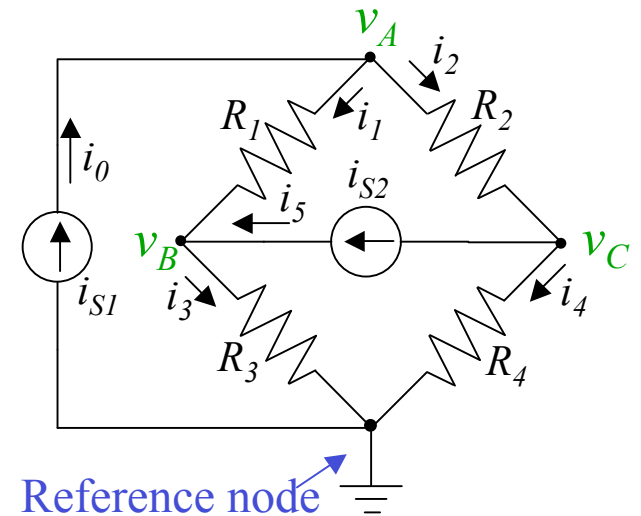Node A: $i_0 - i_1 - i_2 = 0$

Node B: $i_1 - i_3 + i_5 = 0$

Node C: $i_2 - i_4 - i_5 = 0$

Write the element/branch eqns

$i_0 = i_{S0}$ $\qquad$ $i_3 = G_3 v_B$

$i_1 = G_1(v_A - v_B)$ $\qquad$ $i_4 = G_4 v_C$

$i_2 = G_2(v_A - v_C)$ $\qquad$ $i_5 = i_{S2}$

Substitute to get node voltage equations

Node A: $(G_1 + G_2)v_A - G_1 v_B - G_2 v_C = i_{S1}$

Node B: $\qquad -G_1 v_A + (G_1 + G_3)v_B = i_{s2}$

Node C: $\qquad -G_2 v_A + (G_2 + G_4)v_C = -i_{s2}$

$$\begin{pmatrix} G_1+G_2 & -G_1 & -G_2 \\ -G_1 & G_1+G_3 & 0 \\ -G_2 & 0 & G_2+G_4 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$

Solve for v_A, v_B, v_C then i_0, i_1, i_2, i_3, i_4, i_5

Systematic Nodal Analysis

$$\begin{pmatrix} G_1+G_2 & -G_1 & -G_2 \\ -G_1 & G_1+G_3 & 0 \\ -G_2 & 0 & G_2+G_4 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$
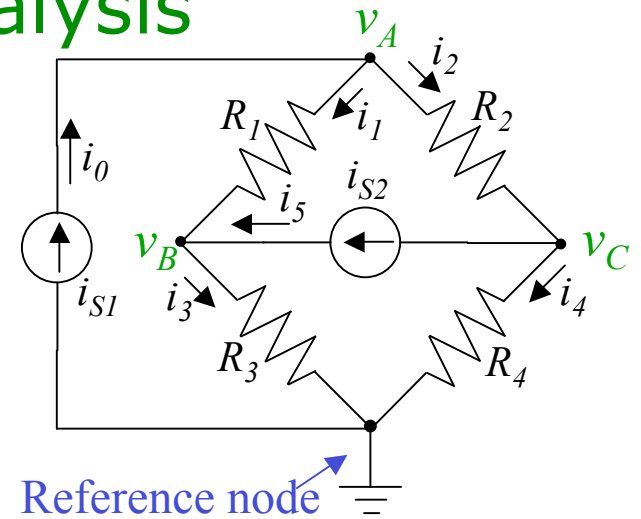
Writing node equations by inspection

Note that the matrix equation looks
just like $\underline{G}\underline{v}=\underline{i}$ for matrix $\underline{G}$ and vector $\underline{v}$ and $\underline{i}$

$\underline{G}$ is symmetric (and non-negative definite)

Diagonal *(i,i)* elements: sum of all conductances connected
to node *i*

Off-diagonal *(i,j)* elements: -conductance between nodes *i*
and *j*

Right-hand side: current sources entering node *i*

There is no equation for the ground node – the column sums
give the conductance to ground

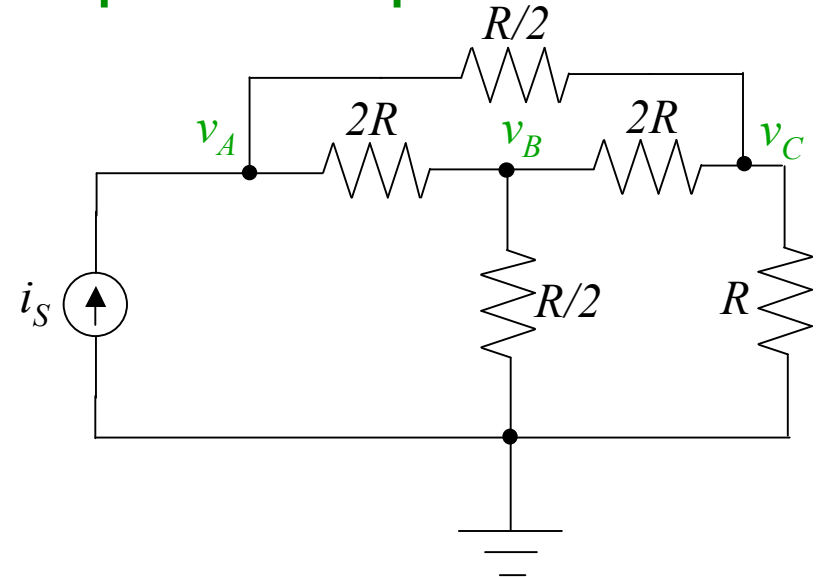# Nodal Analysis Example 3-2 p.74

Node A:

Conductances

$G/2_B + 2G_C = 2.5G$

Source currents entering $= i_S$

Node B:

Conductances

$G/2_A + G/2_C + 2G_{ground} = 3G$ Source currents entering $= 0$

Node C:

Conductances

$2G_A + G/2_B + G_{ground} = 3.5G$ Source currents entering $= 0$

$$\begin{pmatrix} 2.5G & -0.5G & -2G \\ -0.5G & 3G & -0.5G \\ -2G & -0.5G & 3.5G \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_S \\ 0 \\ 0 \end{pmatrix}$$

Nodal Analysis – some points to watch

1. The formulation given is based on KCL with the sum of currents **_leaving_** the node

   $0=i_{total}=G_{AtoB}(v_A-v_B) +G_{AtoC}(v_A-v_C)+...+ G_{AtoGround}v_A+i_{leavingA}$
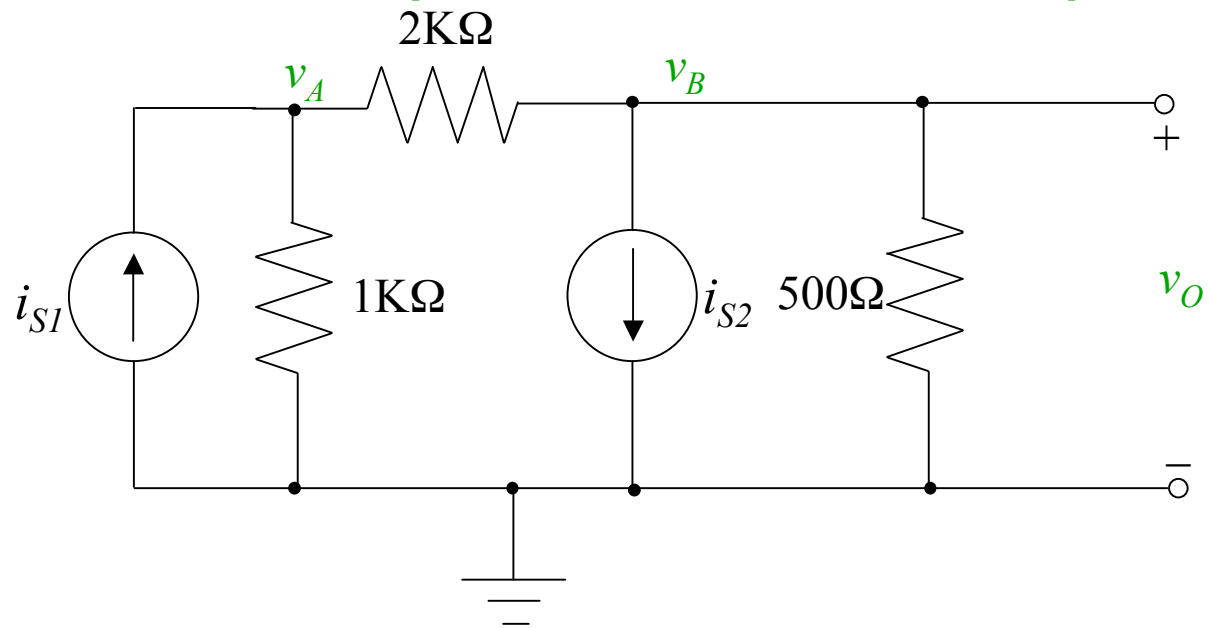
   This yields

   $0=(G_{AtoB}+...+G_{AtoGround})v_A-G_{AtoB}v_B-G_{AtoC}v_C...-i_{enteringA}$

   $(G_{AtoB}+...+G_{AtoGround})v_A-G_{AtoB}v_B-G_{AtoC}v_C...=i_{enteringA}$

2. If in doubt about the sign of the current source, go back to this basic KCL formulation

3. This formulation works for independent current source

   For dependent current sources (introduced later) use your wits

Nodal Analysis Exercise 3-2 p. 72

$$\begin{pmatrix} 1.5 \times 10^{-3} & -0.5 \times 10^{-3} \\ -0.5 \times 10^{-3} & 2.5 \times 10^{-3} \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{pmatrix} i_{S1} \\ -i_{S2} \end{pmatrix}$$

Solve this using standard linear equation solvers

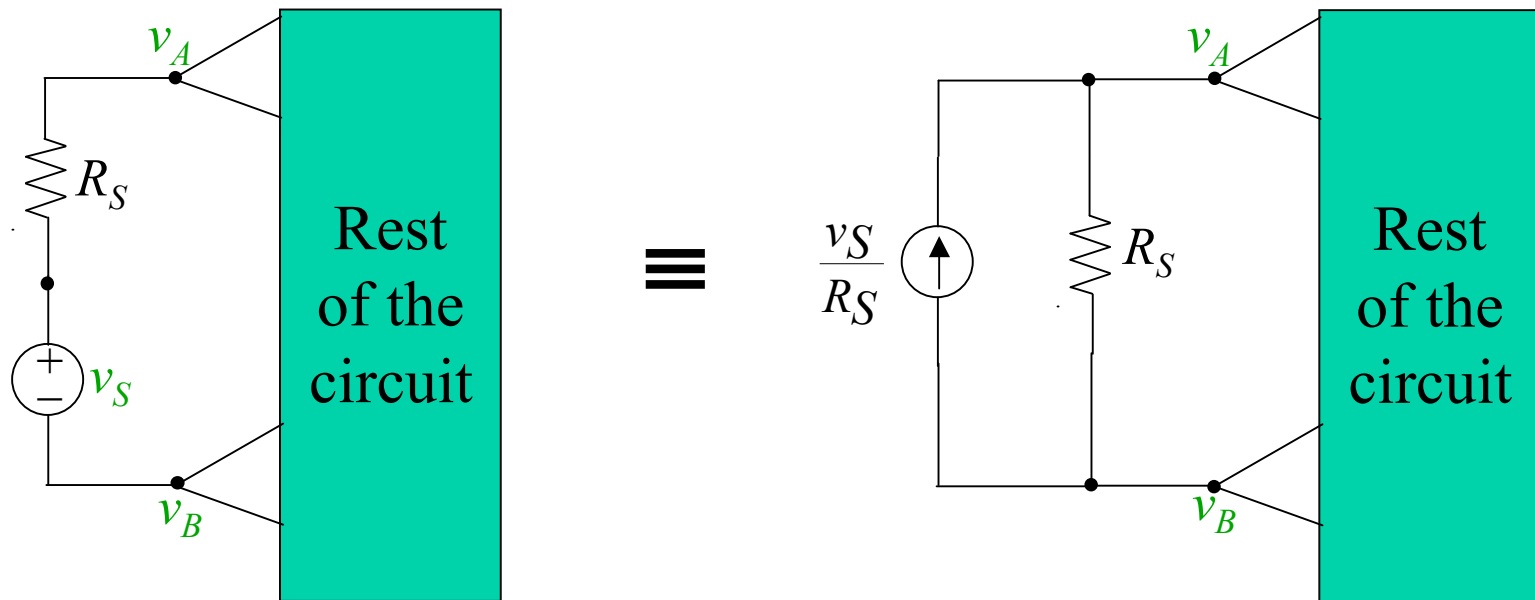Cramer's rule

Gaussian elimination

Matlab

MAE140 Linear Circuits

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Nodal Analysis with Voltage Sources

Current through voltage source is not computable
from voltage across it. We need some tricks!

They actually help us simplify things

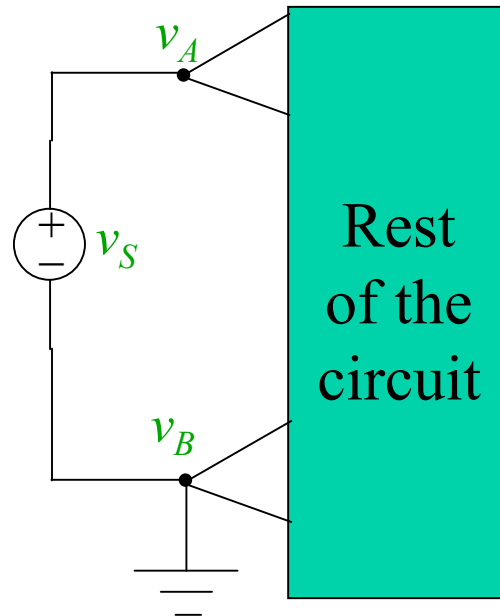Method 1 – source transformation

Then use standard nodal analysis – one less node!

Nodal Analysis with Voltage Sources 2

Method 2 – grounding one node

This removes the v_B variable – simpler analysis
But can be done once per circuit

Nodal Analysis with Voltage Sources 3

Method 3

Create a *supernode*

Act as if A and B were one node

KCL still works for this node

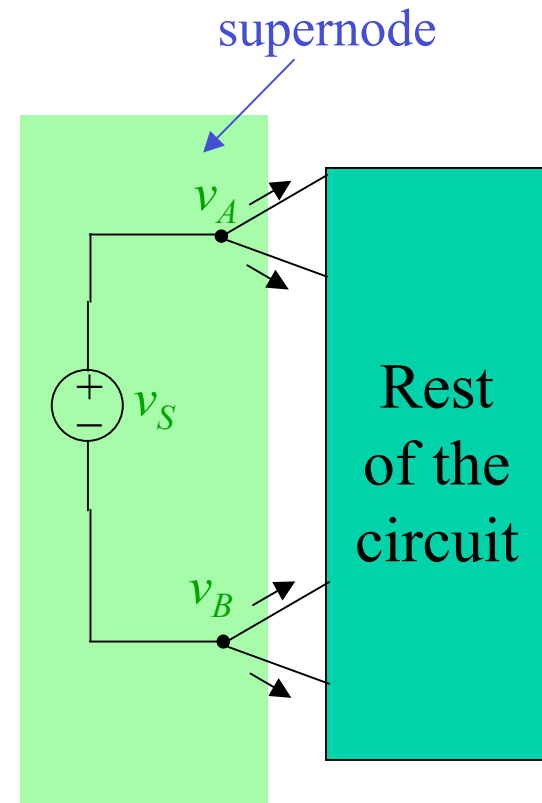Sum of currents entering

supernode box is 0

Write KCL at all N-3 other nodes

N-2 nodes less Ground node

using v_A and v_B as usual

Write one supernode KCL

Add the constraint $v_A - v_B = v_S$

These three methods allow us to deal with all cases

supernode

v_A

v_S

v_B

Rest of the circuit

Nodal Analysis Example 3-4 p. 76
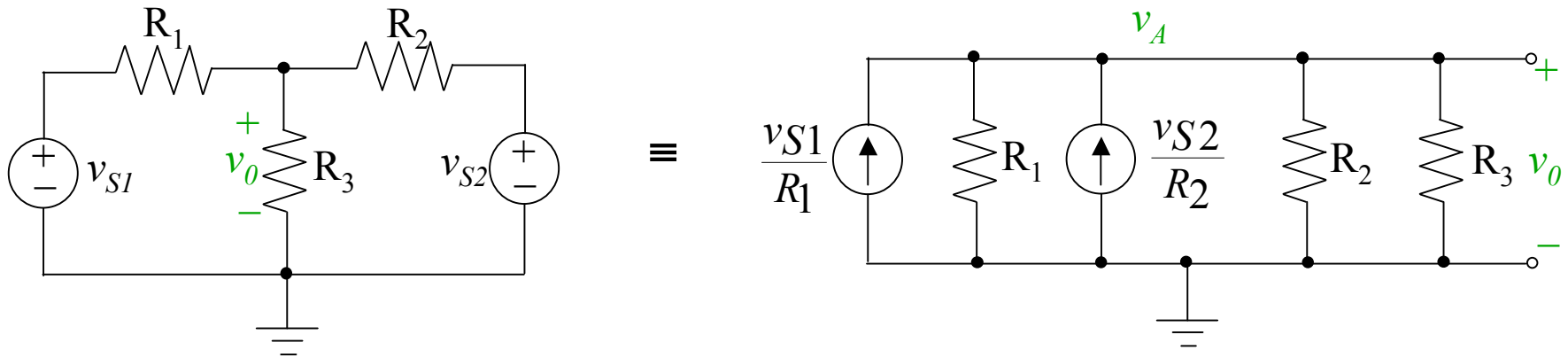
This is method 1 – transform the voltage sources

Applicable since voltage sources appear in series with Rs

Now use nodal analysis with one node, A

$$(G_1 + G_2 + G_3)v_A = G_1 v_{S1} + G_2 v_{S2}$$

$$v_A = \frac{G_1 v_{S1} + G_2 v_{S2}}{G_1 + G_2 + G_3}$$

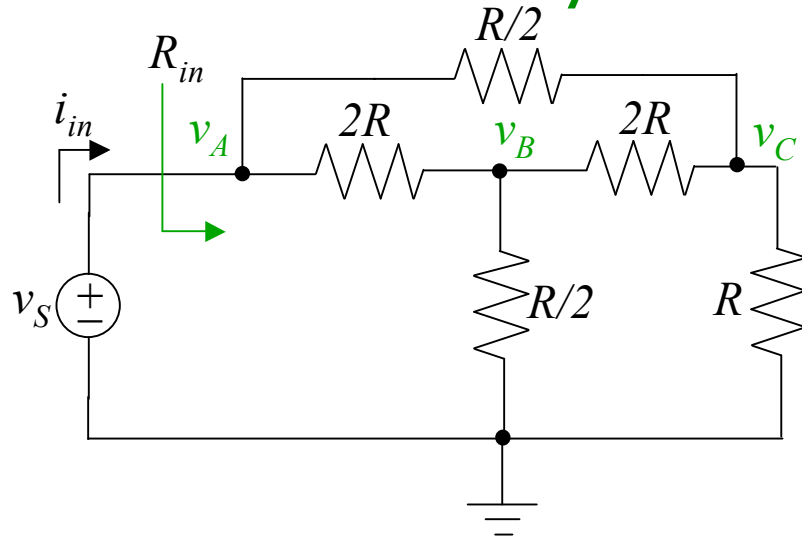# Nodal Analysis Example 3-5 p. 77

What is the circuit input resistance viewed through v_S?

$$v_A = v_S$$

$$-0.5Gv_A + 3Gv_B - 0.5Gv_C = 0$$

$$-2Gv_A - 0.5Gv_B + 3.5v_C = 0$$

Rewrite in terms of v_S, v_B, v_C

This is method 2

$$3Gv_B - 0.5Gv_C = 0.5Gv_S$$

$$-0.5Gv_B + 3.5Gv_C = 2Gv_S$$

$$v_B = \frac{2.75v_S}{10.25}, \quad v_C = \frac{6.25v_S}{10.25}$$

Solve

$$i_{in} = \frac{v_S - v_B}{2R} + \frac{v_S - v_C}{R/2} = \frac{11.75v_S}{10.25R}$$

$$R_{in} = \frac{10.25R}{11.75} = 0.872R$$

Nodal Analysis Example 3-6 p. 78
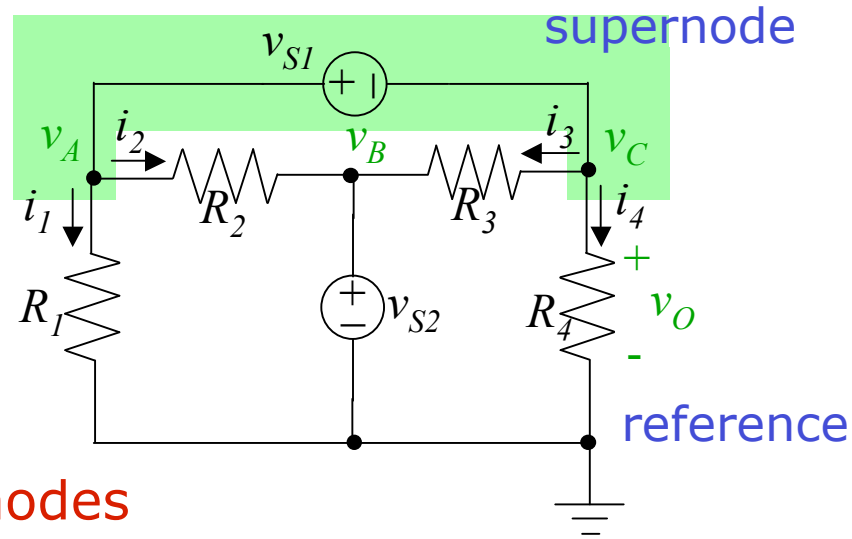
Method 3 – supernodes

KCL for supernode: $i_1 + i_2 + i_3 + i_4 = 0$

Or, using element equations

$$G_1 v_A + G_2(v_A - v_B) + G_3(v_C - v_B) + G_4 v_C = 0$$

Now use $v_B = v_{S2}$

$$(G_1 + G_3)v_A + (G_3 + G_4)v_C = (G_2 + G_3)v_{S2}$$

Other constituent relation

$$v_A - v_C = v_{S1}$$

Two equations in two unknowns   52

Summary of Nodal Analysis

1. Simplify the cct by combining elements in series or parallel

2. Select as reference node the one with most voltage sources connected

3. Label node voltages and supernode voltages – do not label the nodes directly connected to the reference

4. Use KCL to write node equations. Express element currents in terms of node voltages and ICSs

5. Write expressions relating node voltages and IVSs

6. Substitute from Step 5 into equations from Step 4
   Write the equations in standard form

7. Solve using Cramer, gaussian elimination or matlab

Solving sets of linear equations

$$\begin{pmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 6 \end{pmatrix}$$

Cramer's Rule Thomas & Rosa Appendix B pp. A-2 to A-11

$$\Delta = \begin{vmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{vmatrix} = 5 \begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-5) \begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + (-3) \begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix}$$

$$= 5 \left(7 \times 8 - (-3) \times (-2) \right) + 5 \left((-2) \times 8 - (-3) \times (-3) \right) - 3 \left((-2) \times (-2) - 7 \times (-3) \right)$$

$$= 250 - 125 - 75 = 50$$

$$\Delta_1 = \begin{vmatrix} 4 & -2 & -3 \\ -10 & 7 & -2 \\ 6 & -3 & 8 \end{vmatrix} = 4 \begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-10) \begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + 6 \begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix}$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{100}{50} = 2$$

$$= 4 \left(7 \times 8 - (-3) \times (-2) \right) + 10 \left((-2) \times 8 - (-3) \times (-3) \right) + 6 \left((-2) \times (-2) - 7 \times (-3) \right)$$

$$= 200 - 250 + 150 = 100$$

Solving sets of linear equations (contd)

$$\Delta_2 = \begin{vmatrix} 5 & 4 & -3 \\ -5 & -10 & -2 \\ -3 & 6 & 8 \end{vmatrix} = 5\begin{vmatrix} -10 & -2 \\ 6 & 8 \end{vmatrix} - (-5)\begin{vmatrix} 4 & -3 \\ 6 & 8 \end{vmatrix} + (-3)\begin{vmatrix} 4 & -3 \\ -10 & -2 \end{vmatrix}$$

$$=-340+250+114=24$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{24}{50} = 0.48$$

$$\Delta_3 = \begin{vmatrix} 5 & -2 & 4 \\ -5 & 7 & -10 \\ -3 & -3 & 6 \end{vmatrix} = 5\begin{vmatrix} 7 & -10 \\ -3 & 6 \end{vmatrix} - (-5)\begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} + (-3)\begin{vmatrix} -2 & 4 \\ 7 & -10 \end{vmatrix}$$

$$=60-0+24=84$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{84}{50} = 1.68$$

Notes:

This Cramer is not as much fun as Cosmo Kramer in *Seinfeld*

I do not know of any tricks for symmetric matrices

MAE140 Linear Circuits

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Solving Linear Equations – gaussian elimination

$$\begin{pmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 6 \end{pmatrix}$$

Augment matrix ⟹

Row operations only

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ -5 & 7 & -2 & -10 \\ -3 & -3 & 8 & 6 \end{pmatrix}$$

$row_2 + row_1$
$row_3 + row_1 \times 3/5$

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & \dfrac{-21}{5} & \dfrac{31}{5} & \dfrac{42}{5} \end{pmatrix}$$

$row_3 \times 5$

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & -21 & 31 & 42 \end{pmatrix}$$

$row_3 + row_2 \times 21/5$

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & 0 & 10 & \dfrac{84}{5} \end{pmatrix}$$

$row_3 \div 10$

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & 0 & 1 & \dfrac{84}{50} \end{pmatrix}$$

$(row_2 + row_3 \times 5) \div 5$

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 1 & 0 & \dfrac{24}{50} \\ 0 & 0 & 1 & \dfrac{84}{50} \end{pmatrix}$$

$(row_1 + row_2 \times 2 + row_3 \times 3) \div 5$

$$\begin{pmatrix} 1 & 0 & 0 & \dfrac{100}{50} \\ 0 & 1 & 0 & \dfrac{24}{50} \\ 0 & 0 & 1 & \dfrac{84}{50} \end{pmatrix}$$

Solving Linear Equations - matlab

```
A=[5 -2 -3; -5 7 -2; -3 -3 8]

A =   5 -2 -3
     -5  7 -2
     -3 -3  8

B=[4;-10;6]

B =    4
     -10
       6

inv(A)

ans = 1.0000  0.5000  0.5000
      0.9200  0.6200  0.5000
      0.7200  0.4200  0.5000
```

```
inv(A)*B

ans = 2.0000
      0.4800
      1.6800

A\B

ans = 2.0000
      0.4800
      1.6800
```

Mesh Current Analysis
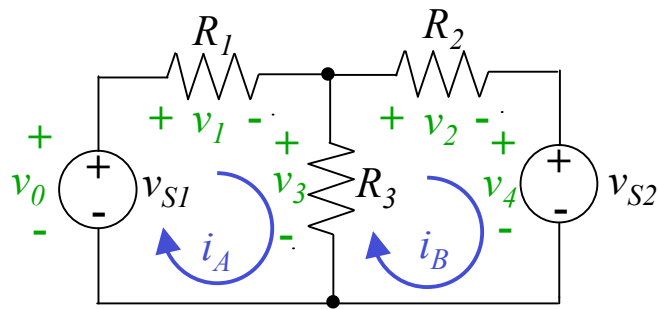
Dual of Nodal Voltage Analysis with KCL

Mesh Current Analysis with KVL

Mesh = loop enclosing no elements

Restricted to Planar Ccts – no crossovers (unless you are really clever)

Key Idea: *If element K is contained in both mesh i and mesh j then its current is $i_k=i_i-i_j$ where we have taken the reference directions as appropriate*

Same old tricks you already know

$$\text{Mesh A: } v_0-v_1-v_3=0 \qquad v_1=R_1 i_A \qquad v_0=v_{S1}$$
$$\text{Mesh B: } v_3-v_2-v_4=0 \qquad v_2=R_2 i_B \qquad v_4=v_{S2}$$
$$v_3=R_3(i_A-i_B)$$

$$(R_1+R_3)i_A-R_3 i_B=v_{S1}$$
$$-R_3 i_A+(R_2+R_3)i_B=-v_{S2}$$

$$\begin{pmatrix} R_1+R_3 & -R_3 \\ -R_3 & R_2+R_3 \end{pmatrix}\begin{pmatrix} i_A \\ i_B \end{pmatrix}=\begin{pmatrix} v_{S1} \\ -v_{S2} \end{pmatrix}$$

Mesh Analysis by inspection $Ri = v_S$

Matrix of Resistances R

Diagonal ii elements: sum of resistances around loop

Off-diagonal ij elements: - resistance shared by loops i and j
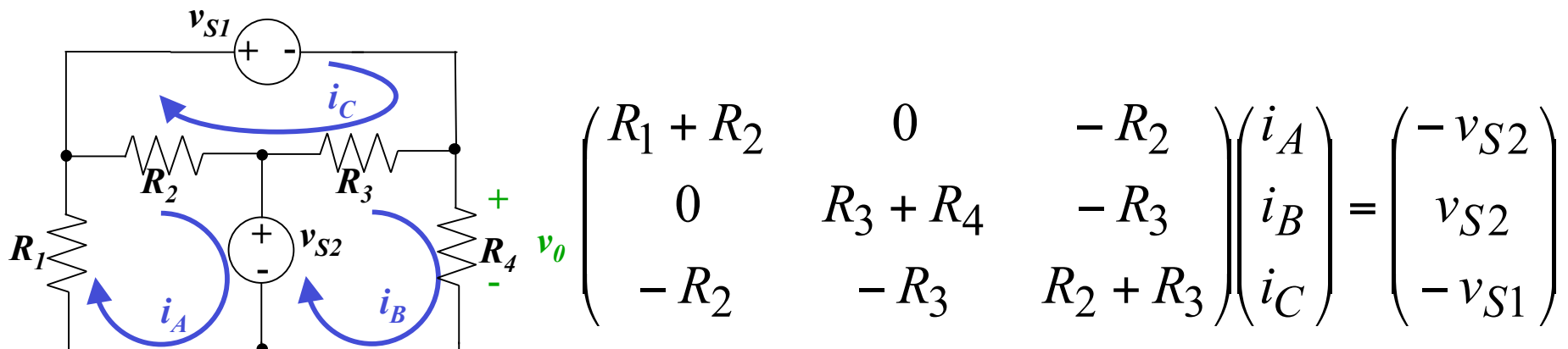
Vector of currents i

As defined by you on your mesh diagram

Voltage source vector v_S

Sum of voltage sources **assisting** the current in your mesh

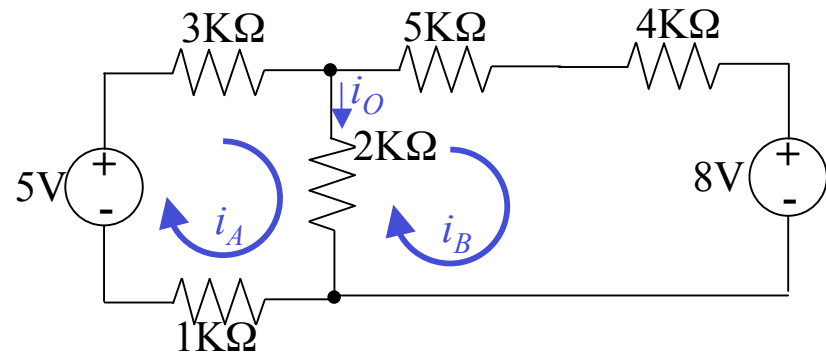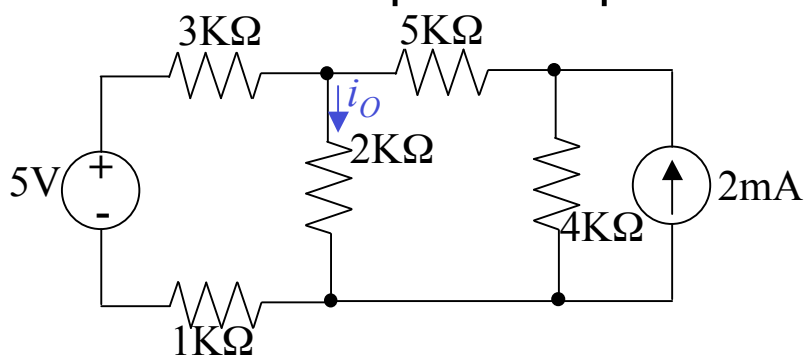If this is hard to fathom, go back to the basic KVL to sort these directions out

$$
\begin{pmatrix} R_1 + R_2 & 0 & -R_2 \\ 0 & R_3 + R_4 & -R_3 \\ -R_2 & -R_3 & R_2 + R_3 \end{pmatrix} \begin{pmatrix} i_A \\ i_B \\ i_C \end{pmatrix} = \begin{pmatrix} -v_{S2} \\ v_{S2} \\ -v_{S1} \end{pmatrix}
$$

Mesh Equations with Current Sources

Duals of tricks for nodal analysis with voltage sources

1. Source transformation to equivalent

T&R Example 3-8 p. 91

$$\begin{pmatrix} 6000 & -2000 \\ -2000 & 11000 \end{pmatrix} \begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$$
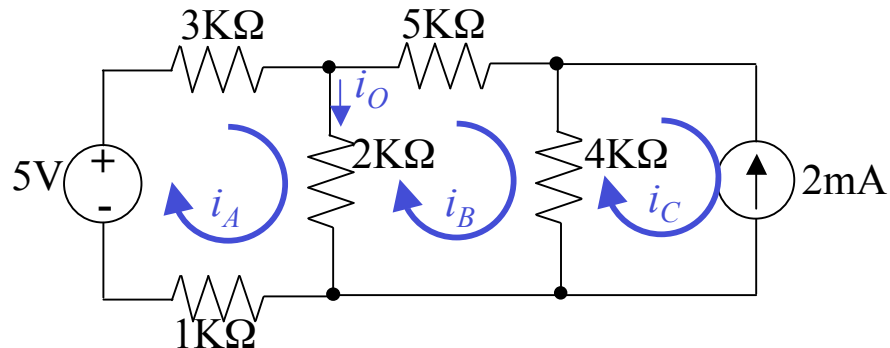
$i_A = 0.6290$ mA
$i_B = -0.6129$ mA
$i_O = i_A - i_B = 1.2419$ mA

Mesh Analysis with ICSs – method 2

Current source belongs to a single mesh

Same example

$$6000i_A - 2000i_B = 5$$

$$-2000i_A + 11000i_B - 4000i_C = 0$$
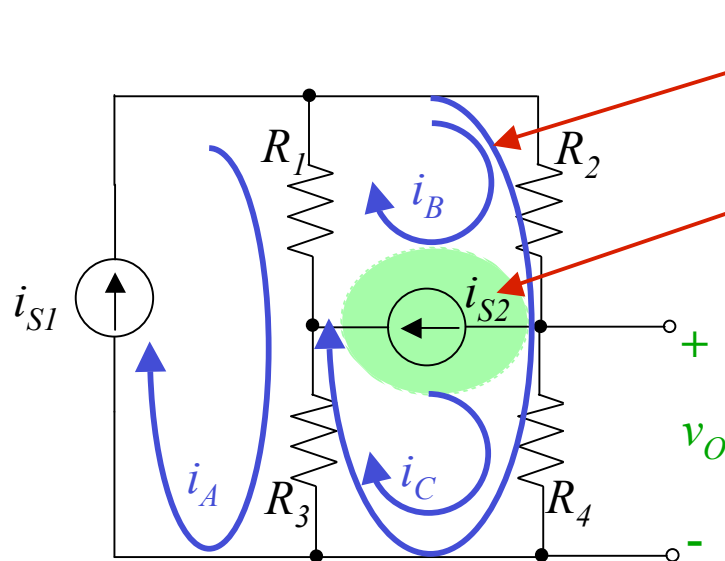
$$i_C = -2\,\text{mA}$$

Same equations!
Same solution

Mesh Analysis with ICSs – Method 3

Supermeshes – easier than supernodes

Current source in more than one mesh and/or not in parallel
with a resistance

1. Create a supermesh by eliminating the whole branch
   involved

2. Resolve the individual currents last

Supermesh

Excluded branch

$$R_1(i_B - i_A) + R_2 i_B + R_4 i_C + R_3(i_C - i_A) = 0$$

$$i_A = i_{S1}$$

$$i_B - i_C = i_{S2}$$

Summary of Mesh Analysis

1. Check if cct is planar or transformable to planar

2. Identify meshes, mesh currents & supermeshes

3. Simplify the cct where possible by combining elements in series or parallel

4. Write KVL for each mesh

5. Include expressions for ICSs

6. Solve for the mesh currents

Linearity & Superposition

Linear cct – modeled by linear elements and independent sources

Linear functions

Homogeneity: $\qquad\qquad f(Kx)=Kf(x)$

Additivity: $\qquad\qquad f(x+y)=f(x)+f(y)$

Superposition –follows from linearity/additivity

Linear cct response to multiple sources is the sum of the responses to each source

1. "Turn off" all independent sources except one and compute cct variables
2. Repeat for each independent source in turn
3. Total value of all cct variables is the sum of the values from all the individual sources
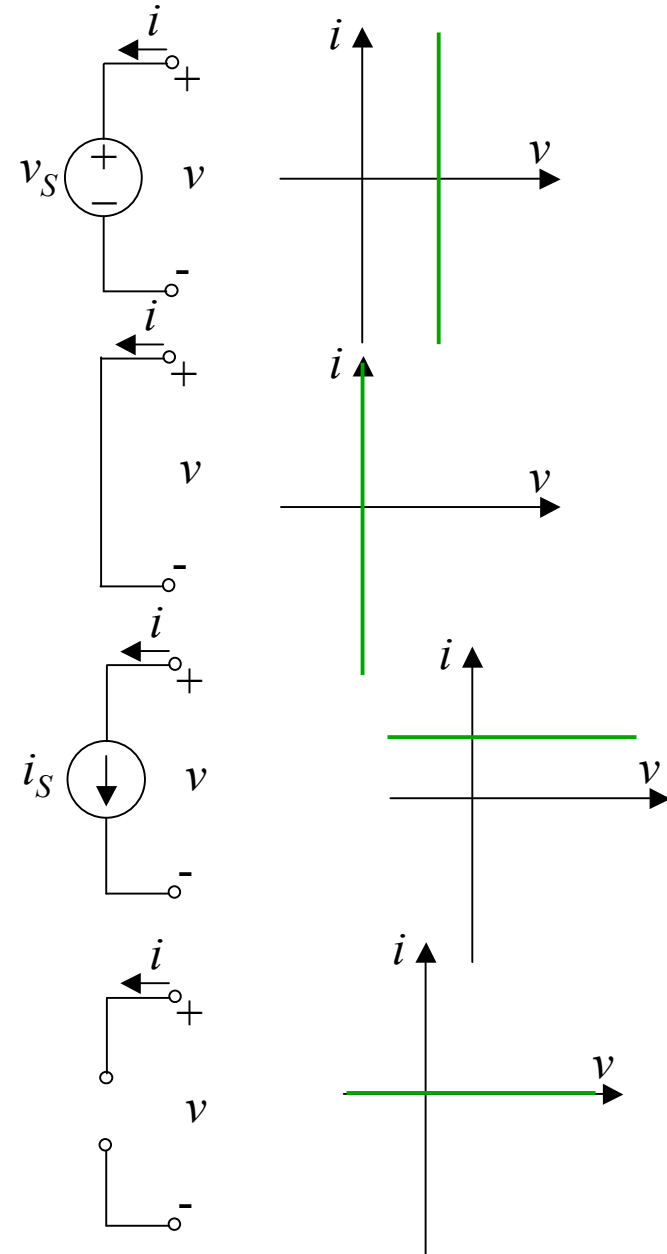
Superposition

Turning off sources

Voltage source

Turned off when $v=0$ for all i

a short circuit

Current source

Turned off when $i=0$ for all v

an open circuit

We have already used this in Thévenin and Norton equiv

Where are we now?

Finished resistive ccts with ICS and IVS

Two analysis techniques – nodal voltage and mesh current

Preference depends on simplicity of the case at hand

The aim has been to develop generalizable techniques for access to analytical tools like matlab

Where to now?

Active ccts with resistive elements – transistors, op-amps

Life starts to get interesting – design introduced

Capacitance and inductance – dynamic ccts

Frequency response – s-domain analysis

Filters