Homework 6 Solution

9-4 Find Laplace transform of $f(t) = 10\left(e^{-2000t} - 2e^{-1000t}\right)u(t)$. Locate poles & zeros!

$$\mathcal{L}\{f(t)\} = 10\,\mathcal{L}\{e^{-2000t}u(t)\} - 20\,\mathcal{L}\{e^{-1000t}u(t)\}$$

$$= 10\frac{1}{s+2000} - 20\frac{1}{s+1000} = \frac{10(s+1000) - 20(s+2000)}{(s+2000)(s+1000)}$$

$$\boxed{\mathcal{L}\{f(t)\} = -10\frac{(s+3000)}{(s+1000)(s+2000)}} \qquad \boxed{\text{poles} = \{-1000, -2000\},\ \text{zeros} = \{-3000, \infty\}}$$

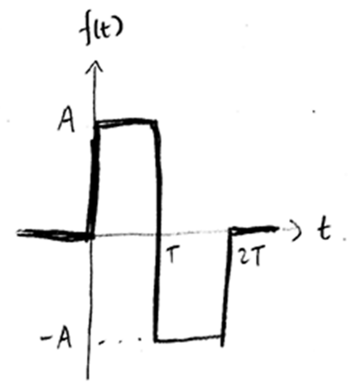
9-9 Find Laplace t. of $f(t) = \delta'(t) + \delta(t) - e^{-t}u(t)$. Locate poles & zeros!

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\delta'(t)\} + \mathcal{L}\{\delta(t)\} - \mathcal{L}\{e^{-t}u(t)\} = s\mathcal{L}\{\delta(t)\} - \delta(0^-) + 1 - \frac{1}{s+1}$$

$\delta(0^-) \rightarrow$ the 0^- is just before 0, so $\delta(0^-) = 0$

$$\hookrightarrow \mathcal{L}\{f(t)\} = s + 1 - \frac{1}{s+1} = \frac{s^2+2s+1-1}{s+1} = \boxed{\frac{s(s+2)}{s+1}} \qquad \boxed{\begin{array}{l}\text{zeros} = 0, -2\\ \text{pole} = -1, \infty\end{array}}$$

9-18



a) $\boxed{f(t) = A\,u(t) - 2A\,u(t-T) + A\,u(t-2T)}$

b) Find $\mathcal{L}\{f(t)\}$!

$$\hookrightarrow \mathcal{L}\{f(t)\} = A\left(\frac{1}{s}\right) - 2A\left(\frac{e^{-Ts}}{s}\right) + A\left(\frac{e^{-2Ts}}{s}\right)$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{A}{s}\left(1 - 2e^{-Ts} + e^{-2Ts}\right)}$$

c) Solve for $\mathcal{L}\{f(t)\}$ using the Laplace transform definition!

$$F\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} A\,u(t)e^{-st}dt - \int_{0^-}^{\infty} 2A\,u(t-T)e^{-st}dt + \int_{0^-}^{\infty} A\,u(t-2T)e^{-st}dt$$

$$= A\int_0^{\infty} e^{-st}dt - 2A\int_T^{\infty} e^{-st}dt + A\int_{2T}^{\infty} e^{-st}dt$$

$$= A\left(\frac{1}{-s}e^{-st}\Big|_{t=0}^{\infty}\right) - 2A\left(\frac{1}{-s}e^{-st}\Big|_{t=T}^{\infty}\right) + A\left(\frac{1}{-s}e^{-st}\Big|_{t=2T}^{\infty}\right)$$

$$= \boxed{A\frac{1}{s} - 2A\frac{e^{-Ts}}{s} + A\frac{e^{-2Ts}}{s}} \longrightarrow \text{the same as b).}$$

9-21 Find inverse Laplace Transform!

a) $F_1(s) = \dfrac{10}{s(s+50)} = \dfrac{A}{s} + \dfrac{B}{s+50} = \dfrac{A(s+50)+Bs}{s(s+50)} = \dfrac{(A+B)s+50A}{s(s+50)}$

$\hookrightarrow \boxed{\begin{array}{l} A = 1/5 \\ B = -1/5 \end{array}} \rightarrow \mathcal{L}^{-1}\{F_1(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1/5}{s} - \dfrac{1/5}{s+50}\right\} = \dfrac{1}{5}u(t) - \dfrac{1}{5}e^{-50t}u(t)$

b) $F_2(s) = \dfrac{s+2}{(s+3)(s+4)} = \dfrac{A}{s+3} + \dfrac{B}{s+4} = \dfrac{A(s+4)+B(s+3)}{(s+3)(s+4)} = \dfrac{(A+B)s+(4A+3B)}{(s+3)(s+4)}$

$\hookrightarrow \begin{array}{l} A+B = 1 \\ 4A+3B = 2 \end{array} \rightarrow \boxed{\begin{array}{l} A = -1 \\ B = 2 \end{array}} \rightarrow \mathcal{L}^{-1}\{F_2(s)\} = \mathcal{L}^{-1}\left\{-\dfrac{1}{s+3} + \dfrac{2}{s+4}\right\}$

$= \left(-e^{-3t} + 2e^{-4t}\right)u(t)$

9-23 Find inverse Laplace!

a) $F_1(s) = \dfrac{50(s+1000)(s+2000)}{(s+500)(s+5000)} = \left(\dfrac{A}{s+500} + \dfrac{B}{s+5000} + C\right)\cdot 50$

$50\left(\dfrac{s^2+3000s+2\cdot10^6}{(s+500)(s+5000)}\right) = 50\left(\dfrac{A(s+5000)+B(s+500)+C(s+500)(s+5000)}{(s+500)(s+5000)}\right)$

$= 50\left(\dfrac{Cs^2+(A+B+5500C)s+(5000A+500B+25\cdot10^5C)}{(s+500)(s+5000)}\right)$

Solving this =

$$\left.\begin{array}{l} C = 1 \\ A+B+5500C = 3000 \\ 5000A+500B+25\cdot10^5C = 2\cdot10^6 \end{array}\right\} \begin{array}{l} A+B = -2500 \\ 5000A+500B = -5\cdot10^5 \end{array} \rightarrow \boxed{\begin{array}{l} A = 500/3 \\ B = -8000/3 \\ C = 1 \end{array}}$$

$\mathcal{L}\{F_1(s)\} = 50\,\mathcal{L}^{-1}\left\{\dfrac{500/3}{s+500} - \dfrac{8000/3}{s+5000} + 1\right\} =$

$= \dfrac{50}{3}\left(500e^{-500t}u(t) - 8000e^{-5000t}u(t)\right) + 50\delta(t)$

b) $F_2(s) = \dfrac{50s^2}{(s+100)(s+500)} = 50\left(\dfrac{A}{s+100} + \dfrac{B}{s+500} + C\right)$

$= 50\left(\dfrac{A(s+500) + B(s+100) + C(s+100)(s+500)}{(s+100)(s+500)}\right)$

$= 50\left(\dfrac{Cs^2 + (A+B+600C)s + (500A+100B+50000C)}{(s+100)(s+500)}\right)$

$$\left.\begin{aligned} C &= 1 \\ A+B+600C &= 0 \\ 500A+100B+50000C &= 0 \end{aligned}\right\} \begin{aligned} A+B &= -600 \\ 500A+100B &= -50000 \end{aligned} \rightarrow \boxed{\begin{aligned} A &= 25 \\ B &= -625 \\ C &= 1 \end{aligned}}$$

$\mathcal{L}^{-1}\{F_2(s)\} = 50\,\mathcal{L}^{-1}\left\{\dfrac{25}{s+100} - \dfrac{625}{s+500} + 1\right\}$

$= 50\left(25e^{-100t}u(t) - 625e^{-500t}u(t) + \delta(t)\right)$

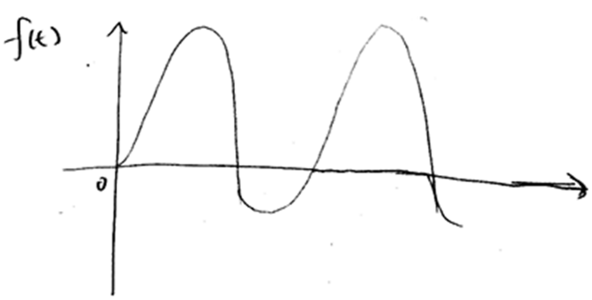
9-26 Find inverse Laplace and sketch w/ $\beta > 0$

a) $F_1(s) = A\dfrac{\beta(s+\beta)}{s(s^2+\beta^2)} = A\beta\left(\dfrac{K_1}{s} + \dfrac{K_2 s}{s^2+\beta^2} + \dfrac{K_3\beta}{s^2+\beta^2}\right)$

$= A\beta\left(\dfrac{K_1(s^2+\beta^2) + K_2 s^2 + K_3 s\beta}{s(s^2+\beta^2)}\right)$

$$\hookrightarrow \begin{aligned} (K_1+K_2) &= 0 \\ (K_3\beta) &= 1 \\ (K_1\beta^2) &= \beta \end{aligned} \rightarrow \boxed{\begin{aligned} K_1 &= 1/\beta \\ K_2 &= -1/\beta \\ K_3 &= 1/\beta \end{aligned}} \rightarrow \boxed{F_1(s) = A\left(\frac{1}{s} - \frac{s}{s^2+\beta^2} + \frac{\beta}{s^2+\beta^2}\right)}$$

$\mathcal{L}^{-1}\{F_1(s)\} = A\,\mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{s}{s^2+\beta^2} + \dfrac{\beta}{s^2+\beta^2}\right\} = A\left(1 - \cos(\beta t) + \sin(\beta t)\right)u(t).$

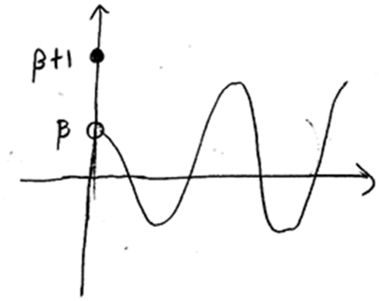


b) $F_2(s) = \dfrac{As(s+\beta)}{s^2+\beta^2} = A\left(\dfrac{s^2+\beta s}{s^2+\beta^2}\right) = A\left(\dfrac{s^2+\beta^2 + (\beta s - \beta^2)}{s^2+\beta^2}\right)$

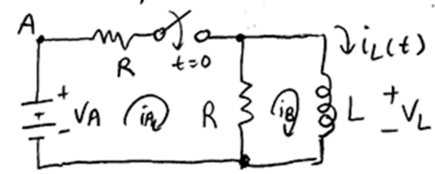
$= A\left(1 + \dfrac{\beta s}{s^2+\beta^2} - \dfrac{\beta \cdot \beta}{s^2+\beta^2}\right)$

↳ $\mathcal{L}^{-1}\{F_2(s)\} = A\,\mathcal{L}^{-1}\left\{1 + \frac{\beta s}{s^2+\beta^2} - \frac{\beta^2}{s^2+\beta^2}\right\}$

$= A\left(\delta(t) + \beta\cos(\beta t)u(t) - \beta\sin(\beta t)u(t)\right)$



9-46 Switch is open for a long time and then closed at $t=0$. $R = 10k\Omega$, $L = 100mH$, $V_A = 24V$,



a) We have $V_L(t) = L\dfrac{di_L(t)}{dt}$

Using mesh current

↳ $-V_A + i_A R + (i_A - i_B)R = 0 \quad (1)$

$(i_B - i_A)R + L\dfrac{di_L(t)}{dt} = 0 \quad (2)$ } Solve for i_B

$i_B = i_L(t)$

(1) → $2R\,i_A = Ri_B + V_A$

$i_A = \dfrac{i_B}{2} + \dfrac{V_A}{2R}$

(2) → $\left(i_B - \dfrac{i_B}{2} - \dfrac{V_A}{2R}\right)R + L\dfrac{di_L(t)}{dt} = 0$

$\dfrac{i_B}{2}R + L\dfrac{di_L(t)}{dt} = \dfrac{V_A}{2}$

Initial condition for $i_L(0) = 0$ because the switch is open → no power.

↳ $\boxed{\dfrac{1}{2}R\,i_L(t) + L\dfrac{di_L(t)}{dt} = \dfrac{V_A}{2}}$

b) s-domain → $\boxed{\dfrac{1}{2}R\,I_L(s) + Ls\,I_L(s) = \dfrac{1}{2}V_A(s)}$

c) $V_A(s) = \mathcal{L}\{V_A \cdot u(t)\} = \dfrac{24V}{s}$

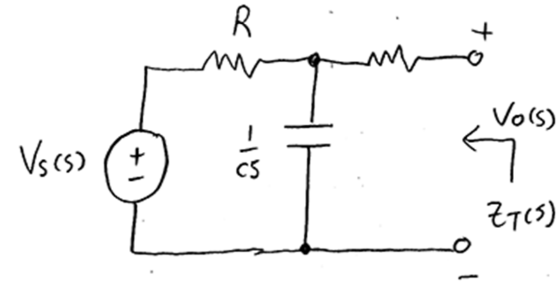
$\left(\dfrac{1}{2}\cdot 10k\Omega + 100mH\,s\right)I_L(s) = \dfrac{24V}{2s}$ → $\boxed{I_L(s) = \dfrac{12V}{s(100mH\,s + 5k\Omega)}}$

Rearrange into $I_L(s) = \dfrac{120}{s(s+50000)} = \dfrac{0.0024}{s} - \dfrac{0.0024}{s+50000}$

d) $\mathcal{L}^{-1}\{I_L(s)\} = \mathcal{L}^{-1}\left\{\frac{0.0024}{s} - \frac{0.0024}{s+50000}\right\}$

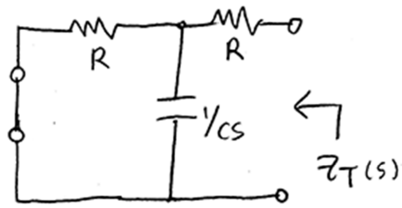
$$= 2.4mA\left(u(t) - e^{-50000t}u(t)\right)$$

10-17 a) Use voltage division to find $V_o(s)$



$\hookrightarrow$ $V_o(s) = \frac{1/cs}{R + 1/cs} V_s(s) = \frac{1}{1+RCs}V_s(s)$

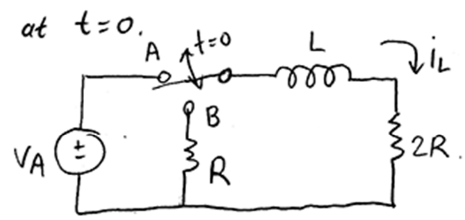
b) Use the lookback method to find Z_T!



$$Z_T(s) = R + \left(\frac{1}{R} + cs\right)^{-1} = R + \left(\frac{R}{1+RCs}\right)$$

$$= \frac{R(1+RCs)+R}{1+RCs} = \frac{RCs + 2R}{RCs + 1}$$

10-26 Switch at B for a long time, then moved to A at $t=0$.

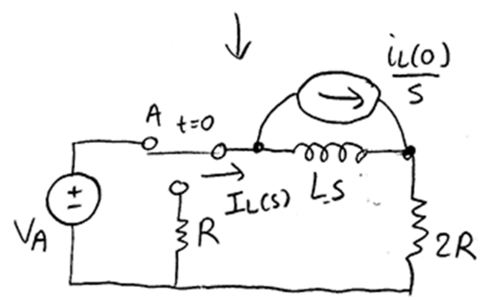


The image indicates that it moved from A to B instead.
Might be a typo, but I will solve for both.
Solve for $I_L(s)$ and $i_L(t)$

$\boxed{\text{B to A}}$

Not connected to power $\rightarrow$ $i_L(0) = 0$

KVL $\rightarrow$ $-V_A(s) + I_L(s)Ls + I_L(s)2R = 0$



$$V_A(s) = \mathcal{L}\{V_A u(t)\} = \frac{V_A}{s}$$

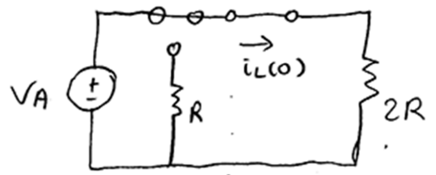
$\hookrightarrow$ $I_L(s) = \frac{V_A/s}{Ls+2R} = \frac{V_A/L}{(s+2R/L)s} = \frac{V_A/2R}{s} - \frac{V_A/2R}{s+\frac{2R}{L}}$

$$i_L(t) = \mathcal{L}^{-1}\{I_L(s)\} = \frac{V_A}{2R}\left(u(t) - e^{-\frac{2R}{L}t}u(t)\right)$$

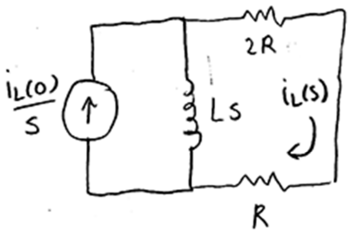
A to B

When connected for a long time, inductance becomes like a short circuit.

↳ $i_L(0) = V_A/2R$

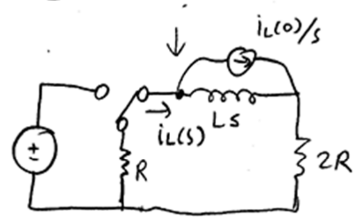


Do circuit reduction and current division



$$I_L(s) = \left(\frac{1/3R}{1/Ls + 1/3R}\right)\frac{i_L(0)}{s}$$

$$= \left(\frac{s}{\frac{3R}{L} + s}\right)\frac{V_A}{2Rs}$$



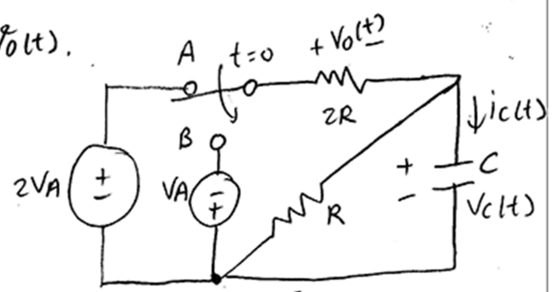
↳ $\boxed{I_L(s) = \frac{V_A}{2R}\left(\frac{1}{s + 3R/L}\right)}$

Solve for $i_L(t)$

↳ $i_L(t) = \mathcal{L}^{-1}\{I_L(s)\} = \underline{\underline{\frac{V_A}{2R}\left(e^{-3R/L\,t}\,u(t)\right)}}$

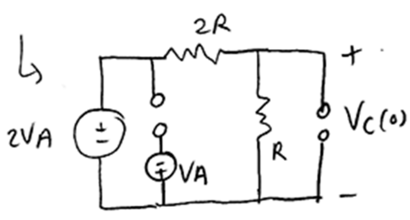
10-27

Pos A to pos B. Solve for $I_C(s)$, $i_C(t)$, $V_O(s)$, $V_O(t)$.



Held for a long time, capacitor becomes like an open circuit.

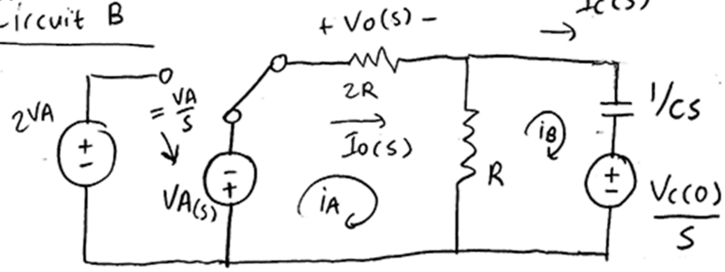
↳



Voltage division

↳ $V_C(0) = \frac{R}{2R+R}\cdot 2V_A = \frac{2}{3}V_A$

Circuit B



Solve for i_A, i_B using mesh current.

$I_O(s) = i_A$

$I_C(s) = i_B$

mesh current

$i_A \rightarrow \quad V_A(s) + (2R+R)\, i_A - R\, i_B = 0$

$\frac{V_A}{s} + 3R\, i_A - R\, i_B = 0$

$i_B \rightarrow \quad \left(R + \frac{1}{Cs}\right) i_B - R\, i_A + \frac{V_C(0)}{s} = 0$

$-R\, i_A + \left(R + \frac{1}{Cs}\right) i_B + \frac{2V_A}{3s} = 0$

$$\begin{bmatrix} 3R & -R \\ -R & R + \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} -\frac{V_A}{s} \\ -\frac{2V_A}{3s} \end{bmatrix}$$

Solve in Matlab $\rightarrow$

$$\boxed{\begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} I_O(s) \\ I_C(s) \end{bmatrix} = \begin{bmatrix} -\frac{V_A}{3Rs}\left(\frac{3+5CRs}{3+2CRs}\right) \\ -\frac{3V_A C}{2CRs+3} \end{bmatrix}}$$

Then $i_C(t) = \mathcal{L}^{-1}\{I_C(s)\} = \mathcal{L}^{-1}\left\{-\frac{3V_A C}{2RC}\left(\frac{1}{s + 3/2RC}\right)\right\}$

$= -\frac{3V_A}{2R} e^{-\frac{3}{2RC}t}\, u(t)$

$V_O(s) = 2R \cdot I_O(s) = -\frac{2V_A}{3s}\left(\frac{3+5CRs}{3+2CRs}\right) = -\frac{2V_A}{3}\left(\frac{K_1}{s} + \frac{K_2}{3+2CRs}\right)$

$= -\frac{2V_A}{3}\left(\frac{3K_1 + 2CRsK_1 + K_2 s}{s(3+2CRs)}\right) \rightarrow \quad K_1 = 1$

$K_2 = 3CR$

$\hookrightarrow \boxed{V_O(s) = -\frac{2V_A}{3}\left(\frac{1}{s} + \frac{3CR}{3+2CRs}\right)}$

$V_O(t) = \mathcal{L}^{-1}\{V_O(s)\} = -\frac{2V_A}{3}\left(u(t) + \frac{3}{2} e^{-\frac{3}{2CR}t}\, u(t)\right)$