s-Domain Circuit Analysis

Operate directly in the *s*-domain with capacitors, inductors and resistors

Key feature – linearity is preserved

Ccts described by ODEs and their ICs

Order equals number of C plus number of L

Element-by-element and source transformation

Nodal or mesh analysis for *s*-domain cct variables

Solution via Inverse Laplace Transform

Why?

1. Easier than ODEs
2. Easier to perform engineering design
3. Frequency response ideas - filtering

Element Transformations

Voltage source

Time domain

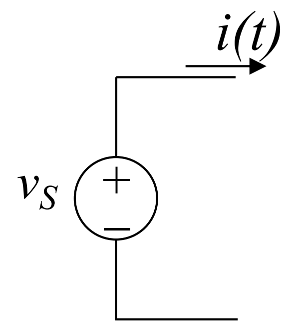
$$v(t) = v_S(t)$$

$i(t)$ = depends on cct

Transform domain

$$V(s) = V_S(s) = \mathcal{L}(v_S(t))$$

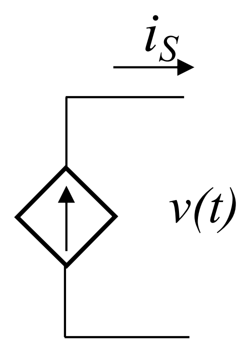
$I(s) = \mathcal{L}(i(t))$ depends on cct



Current source

$$I(s) = \mathcal{L}(i_S(t))$$

$V(s) = \mathcal{L}(v(t))$ depends on cct



Element Transformations contd

Controlled sources

$$v_1(t) = \mu v_2(t) \quad \Leftrightarrow \quad V_1(s) = \mu V_2(s)$$
$$i_1(t) = \beta i_2(t) \quad \Leftrightarrow \quad I_1(s) = \beta I_2(s)$$
$$v_1(t) = r i_2(t) \quad \Leftrightarrow \quad V_1(s) = r I_2(s)$$
$$i_1(t) = g v_2(t) \quad \Leftrightarrow \quad I_1(s) = g V_2(s)$$

Short cct, open cct, OpAmp relations

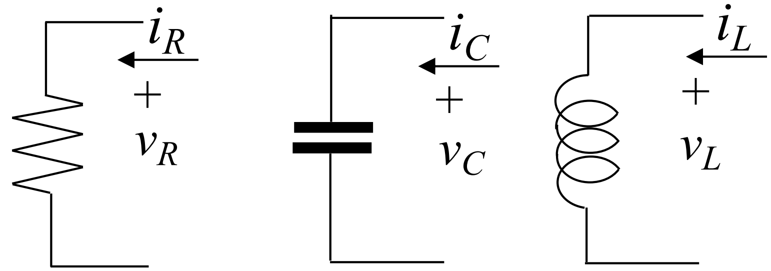
$$v_{SC}(t) = 0 \quad \Leftrightarrow \quad V_{SC}(s) = 0$$
$$i_{OC}(t) = 0 \quad \Leftrightarrow \quad I_{OC}(s) = 0$$
$$v_N(t) = v_P(t) \quad \Leftrightarrow \quad V_N(s) = V_P(s)$$

Sources and active devices behave identically

Constraints expressed between transformed variables

This all hinges on uniqueness of Laplace Transforms and linearity

Element Transformations contd



Resistors

$$v_R(t) = R i_R(t) \qquad i_R(t) = G v_R(t)$$

$$V_R(s) = R I_R(s) \qquad I_R(s) = G V_R(s)$$

$$Z_R(s) = R \qquad Y_R(s) = \frac{1}{R}$$

Capacitors

$$i_C(t) = C\frac{dv_C(t)}{dt} \qquad v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0)$$

$$I_C(s) = sCV_C(s) - Cv_C(0-) \qquad V_C(s) = \frac{1}{sC}I_C(s) + \frac{v_C(0)}{s}$$

Inductors

$$Z_C(s) = \frac{1}{sC} \qquad Y_C(s) = sC$$

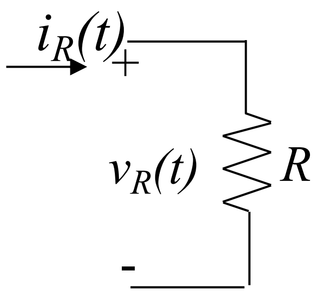
$$v_L(t) = L\frac{di_L(t)}{dt} \qquad i_L(t) = \frac{1}{L}\int_0^t v_L(\tau)d\tau + i_L(0)$$

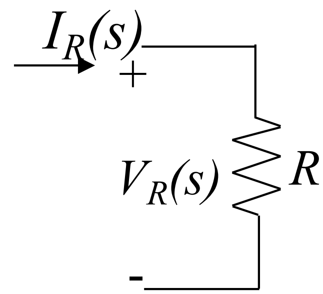
$$V_L(s) = sLI_L(s) - Li_L(0) \qquad I_L(s) = \frac{1}{sL}V_L(s) + \frac{i_L(0)}{s}$$

$$Z_L(s) = sL \qquad Y_L(s) = \frac{1}{sL}$$

Element Transformations contd

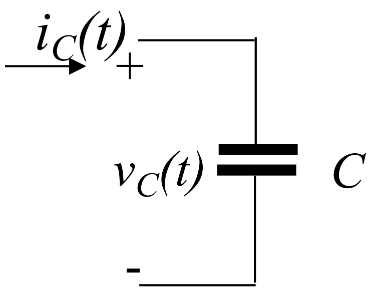
Resistor

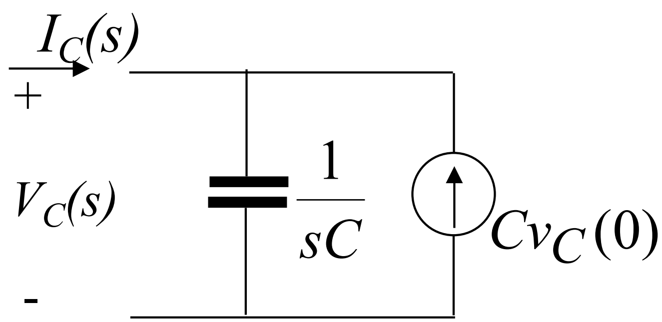


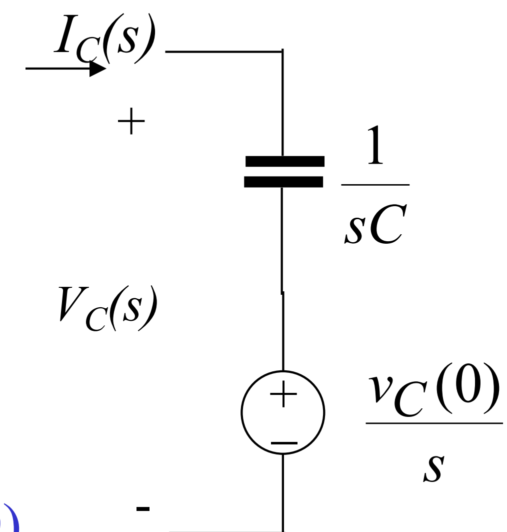


$$V_R(s) = RI_R(s)$$

Capacitor







$$I_C(s) = sCV_C(s) - Cv_C(0-)$$

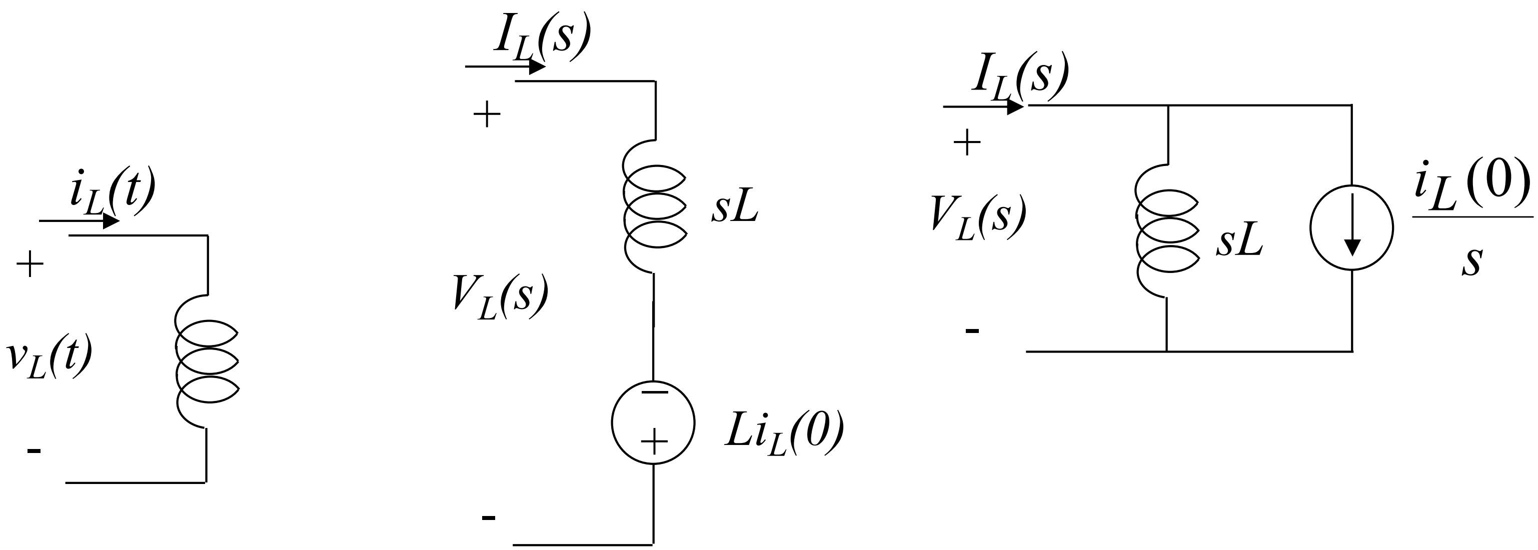
$$V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s}$$

Note the source transformation rules apply!

Element Transformations contd

Inductors

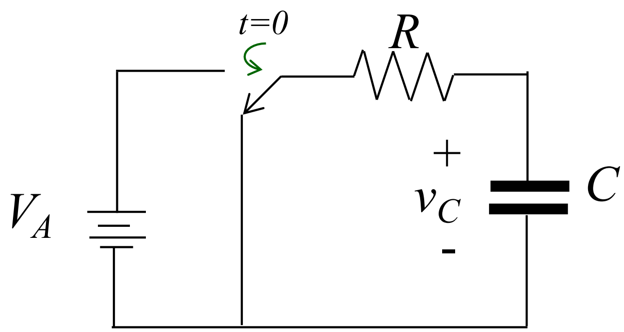
$$V_L(s) = sLI_L(s) - Li_L(0) \qquad I_L(s) = \frac{1}{sL}V_L(s) + \frac{i_L(0)}{s}$$

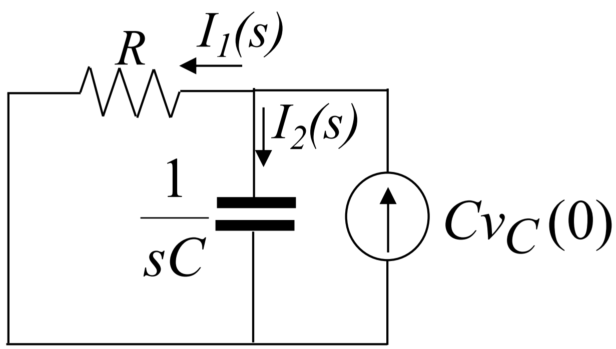


Example 10-1, T&R, 5th ed, p 456

RC cct behavior

Switch in place since t=-∞, closed at t=0. Solve for $v_C(t)$.





Initial conditions $v_C(0) = V_A$

s-domain solution using nodal analysis

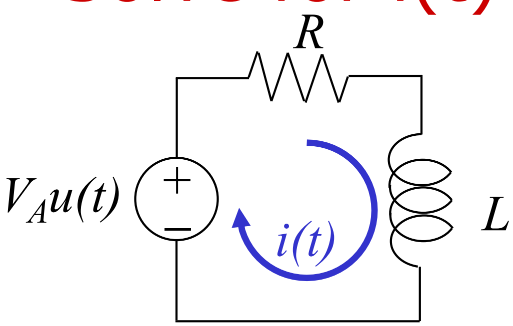
$$I_1(s) = \frac{V_C(s)}{R} \qquad I_2(s) = \frac{V_C(s)}{1/sC} = sCV_C(s)$$

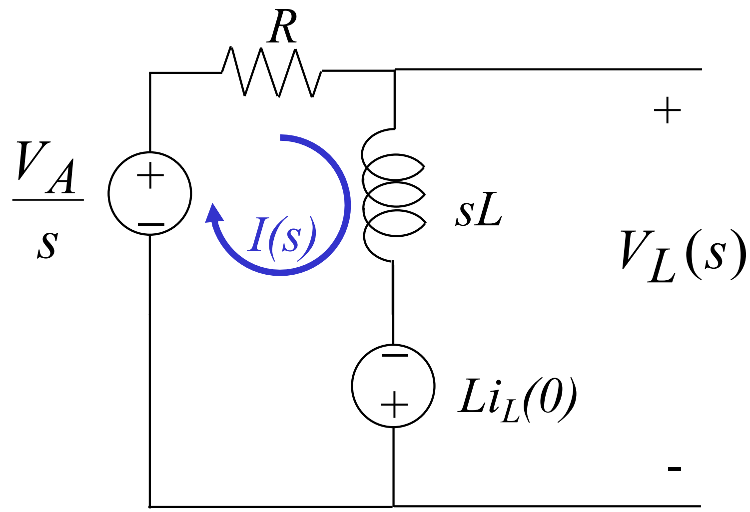
t-domain solution via inverse Laplace transform

$$V_C(s) = \frac{V_A}{s + \frac{1}{RC}} \qquad v_c(t) = V_A e^{-t/RC} u(t)$$

Example 10-2 T&R, 5th ed, p 457

Solve for i(t)





KVL around loop

$$\frac{V_A}{s} - (R + sL)I(s) + Li_L(0) = 0$$

Solve

$$I(s) = \frac{V_A/L}{s\left(s + R/L\right)} + \frac{i_L(0)}{s + R/L} = \frac{V_A/R}{s} + \frac{\left(i_L(0) - V_A/R\right)}{s + R/L}$$

Invert

$$i(t) = \left[\frac{V_A}{R} - \frac{V_A}{R}e^{-Rt/L} + i_L(0)e^{-Rt/L}\right]u(t) \text{ Amps}$$

Impedance and Admittance

Impedance (Z) is the *s*-domain proportionality factor relating the transform of the voltage across a two-terminal element to the transform of the current through the element with all initial conditions zero

$$V(s) = Z(s)I(s)$$

Admittance (Y) is the *s*-domain proportionality factor relating the transform of the current through a two-terminal element to the transform of the voltage across the element with initial conditions zero

$$I(s) = Y(s)V(s)$$

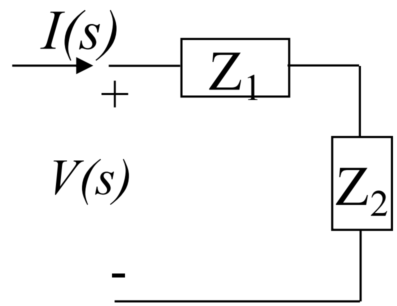
Impedance is like resistance

Admittance is like conductance

Circuit Analysis in s-Domain

Basic rules

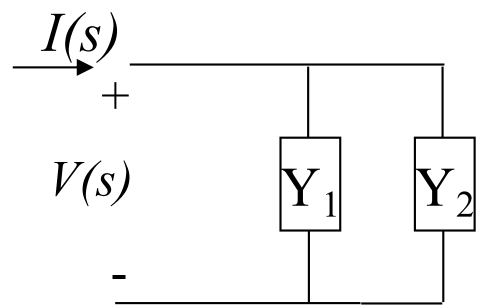
The equivalent impedance $Z_{eq}(s)$ of two impedances $Z_1(s)$ and $Z_2(s)$ in series is $Z_{eq}(s) = Z_1(s) + Z_2(s)$



Same current flows

$$V(s) = Z_1(s)I(s) + Z_2(s)I(s) = Z_{eq}(s)I(s)$$

The equivalent admittance $Y_{eq}(s)$ of two admittances $Y_1(s)$ and $Y_2(s)$ in parallel is $Y_{eq}(s) = Y_1(s) + Y_2(s)$

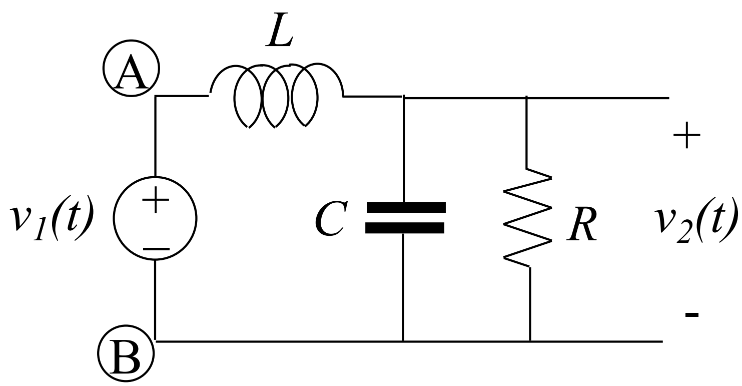


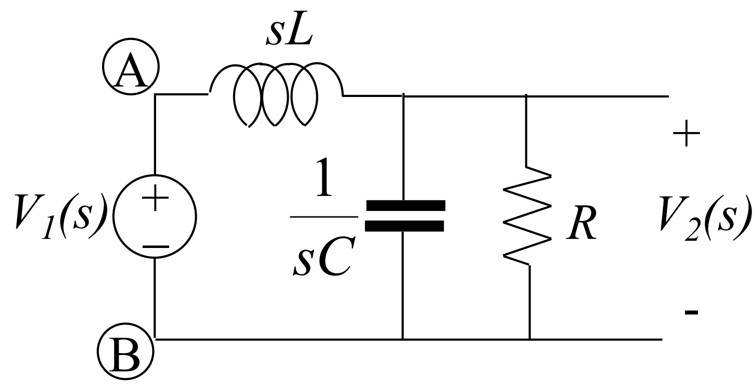
Same voltage

$$I(s) = Y_1(s)V(s) + Y_2(s)V(s) = Y_{eq}(s)V(s)$$

Example 10-3 T&R, 5th ed, p 461

Find $Z_{AB}(s)$ and then find $V_2(s)$ by voltage division



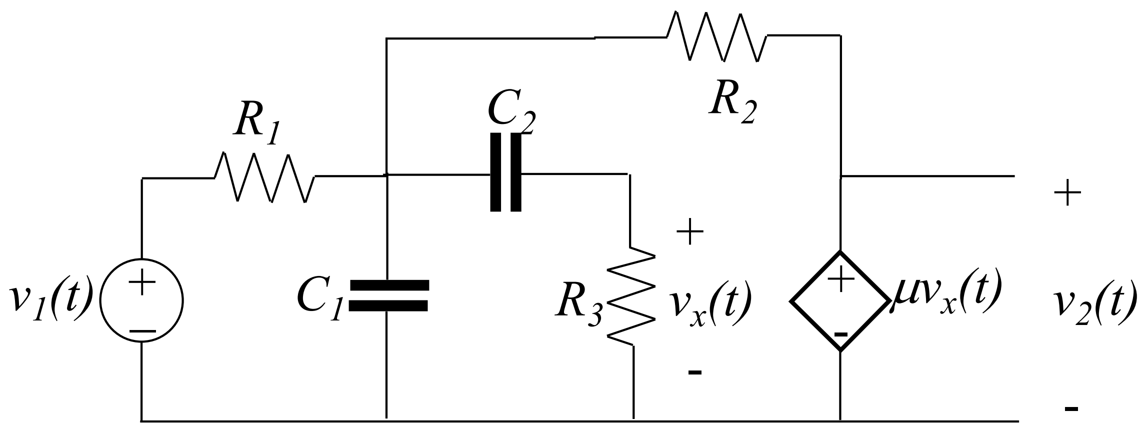


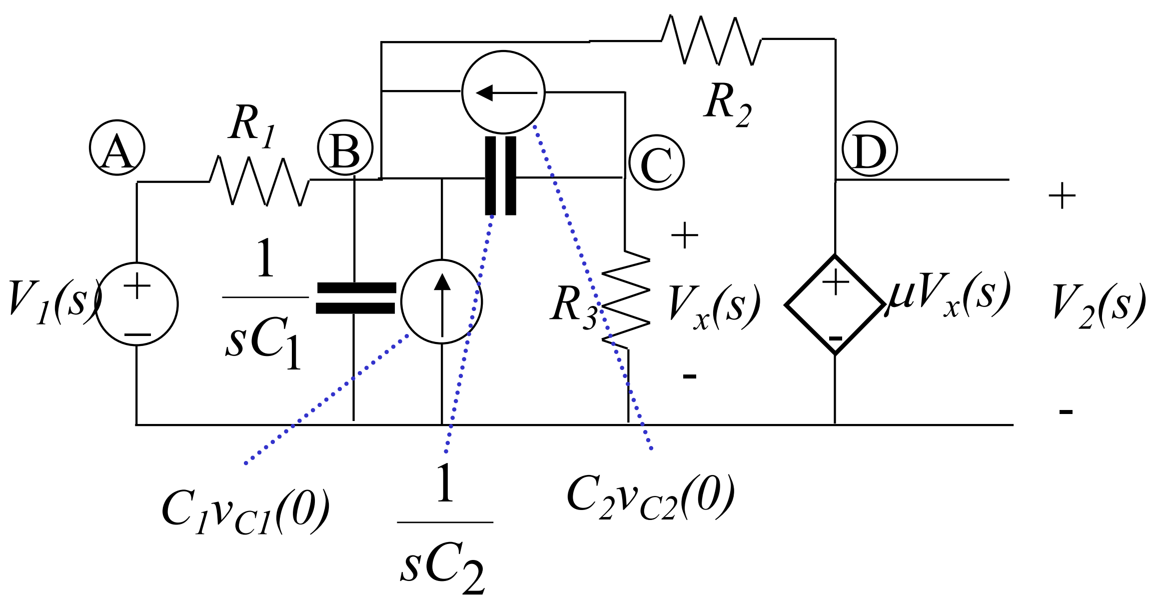
$$Z_{eq}(s) = sL + R \| \frac{1}{sC} = sL + \frac{1}{\frac{1}{R} + sC} = \frac{RLCs^2 + Ls + R}{RCs + 1}\,\Omega$$

$$V_2(s) = \left[\frac{Z_1(s)}{Z_{eq}(s)}\right] V_1(s) = \left[\frac{R}{RLCs^2 + sL + R}\right] V_1(s)$$

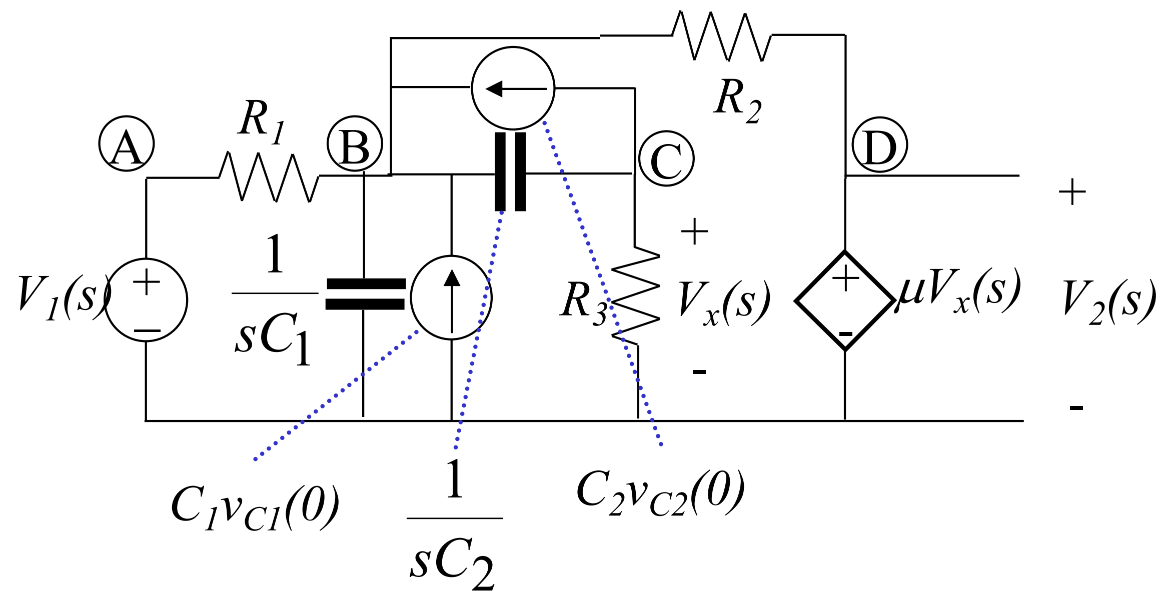
Example

Formulate node voltage equations in the *s*-domain





Example contd



Node A: $V_A(s) = V_1(s)$

Node D: $V_D(s) = \mu V_x(s) = \mu V_C(s)$

Node B:

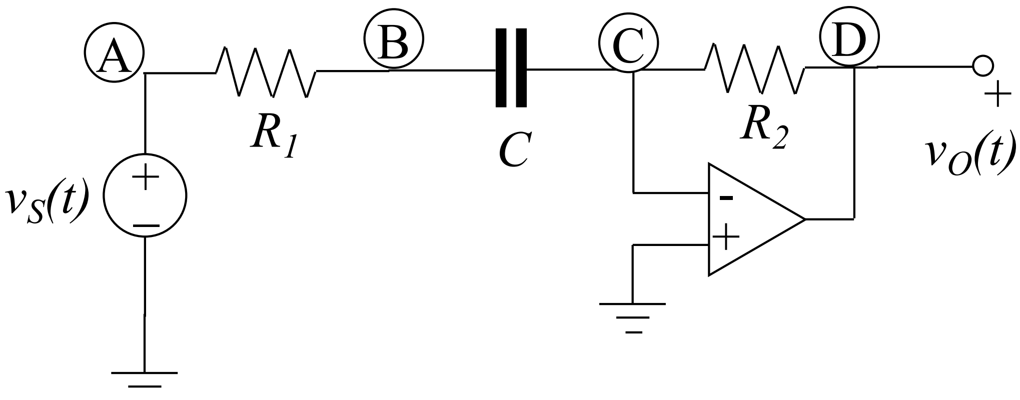
$$\frac{V_B(s) - V_A(s)}{R_1} + \frac{V_B(s) - V_D(s)}{R_2} + \frac{V_B(s)}{1/sC_1} + \frac{V_B(s) - V_C(s)}{1/sC_2} - C_1 v_{C1}(0) - C_2 v_{C2}(0) = 0$$

Node C:

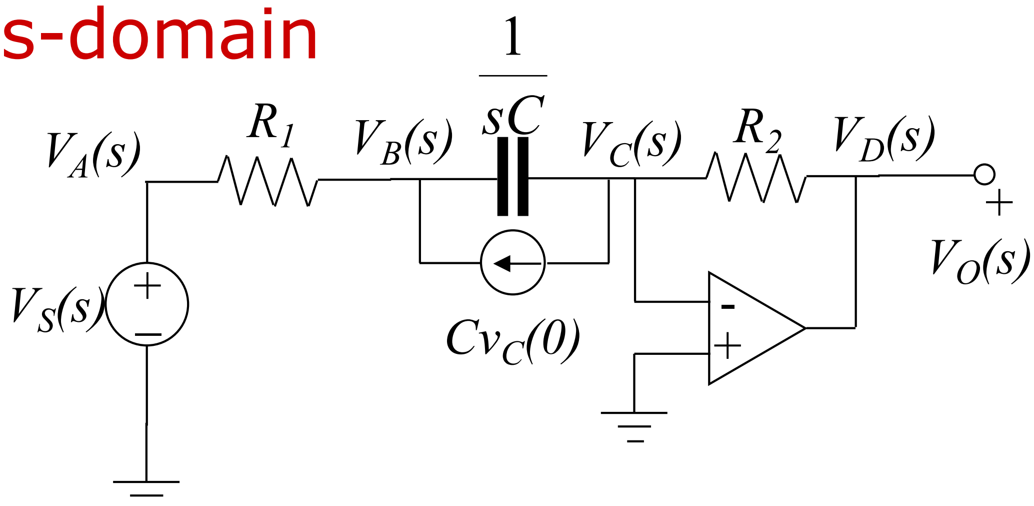
$$-sC_2 V_B(s) + \left[sC_2 + G_3\right] V_C(s) = -C_2 v_{C2}(0)$$

Example

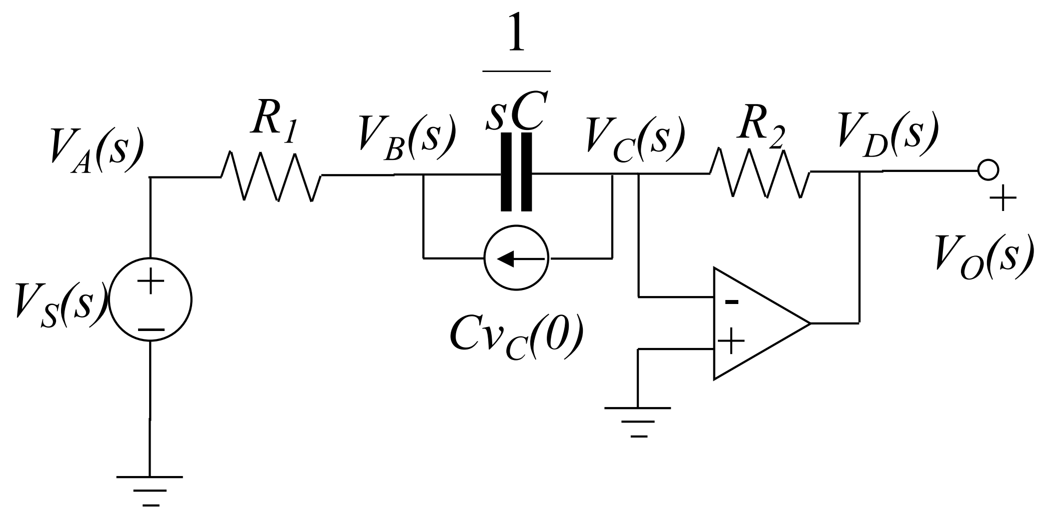
Find $v_O(t)$ when $v_S(t)$ is a unit step $u(t)$ and $v_C(0)=0$



Convert to s-domain



Example



Nodal Analysis

Node A: $V_A(s) = V_S(s)$

Node D: $V_D(s) = V_O(s)$

Node C: $V_C(s) = 0$

Node B: $(G_1 + sC)V_B(s) - G_1 V_S(s) = Cv_C(0)$

Node C KCL: $-sCV_B(s) - G_2 V_O(s) = -Cv_C(0)$

Solve for $V_O(s)$

$$V_O(s) = -\left[\frac{sG_1C/G_2}{G_1 + sC}\right]V_S(s) = -\left[\frac{R_2}{R_1} \times \frac{s}{s + 1/R_1C}\right]V_S(s)$$

$$= -\left[\frac{R_2}{R_1} \times \frac{s}{s + 1/R_1C}\right]\frac{1}{s} = \frac{-R_2}{R_1} \times \frac{1}{s + 1/R_1C}$$

Invert LT

$$v_O(t) = \frac{-R_2}{R_1} e^{-\frac{t}{R_1C}} u(t)$$

Superposition in s-domain ccts

The *s*-domain response of a cct can be found as the sum of two responses

1. The zero-input response caused by initial condition sources, with all external inputs turned off
2. The zero-state response caused by the external sources, with initial condition sources set to zero

   Linearity and superposition

Another subdivision of responses

1. Natural response – the general solution

   Response representing the natural modes (poles) of cct

2. Forced response – the particular solution

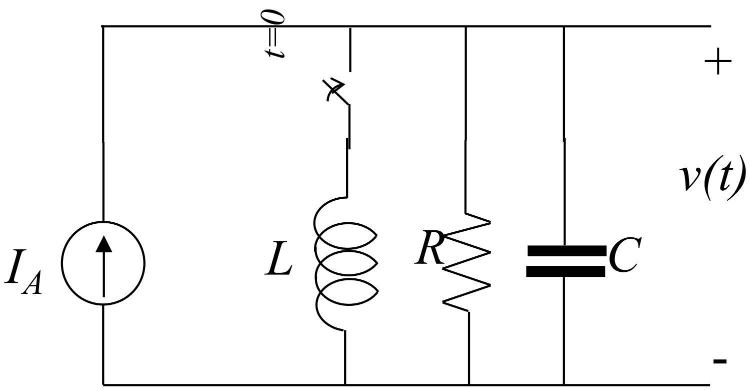
   Response containing modes due to the input

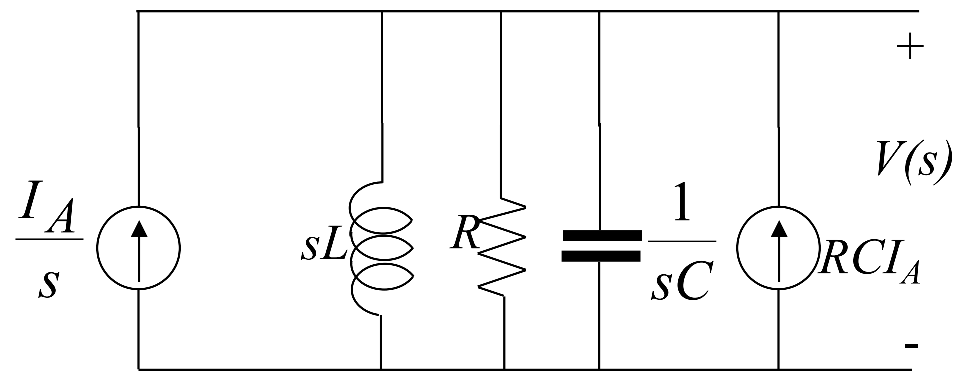
Example 10-6, T&R, 5th ed, p 466

The switch has been open for a long time and is closed at t=0.

Find the zero-state and zero-input components of V(*s*)

Find v(t) for I_A=1mA, L=2H, R=1.5KΩ, C=1/6 μF





$$Z_{eq}(s) = \frac{1}{\frac{1}{sL} + \frac{1}{R} + sC} = \frac{RLs}{RLCs^2 + Ls + R}$$

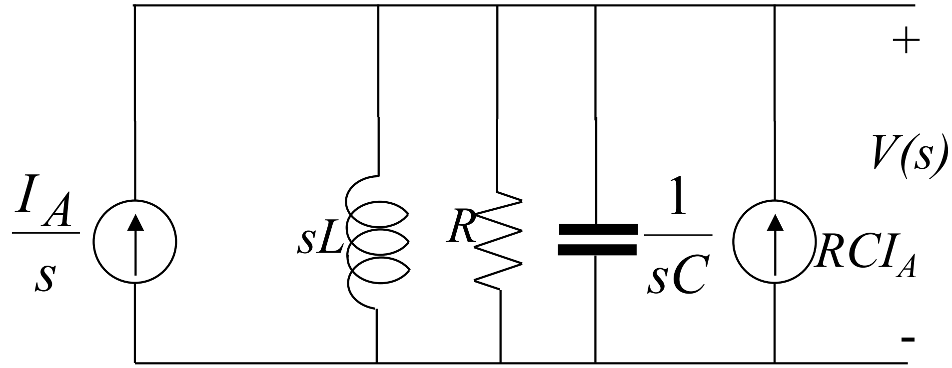
$$V_{zs}(s) = Z_{eq}(s)\frac{I_A}{s} = \frac{I_A/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V_{zi}(s) = Z_{eq}(s)RCI_A = \frac{RI_A s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Example 10-6 contd

$$V_{zs}(s) = Z_{eq}(s)\frac{I_A}{s} = \frac{I_A/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V_{zi}(s) = Z_{eq}(s)RCI_A = \frac{RI_A s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$



Substitute values

$$V_{zs}(s) = \frac{6000}{(s+1000)(s+3000)} = \frac{3}{s+1000} + \frac{-3}{s+3000}$$

$$v_{zs}(t) = \left[3e^{-1000t} - 3e^{-3000t}\right]u(t)$$

$$V_{zi}(s) = \frac{1.5s}{(s+1000)(s+3000)} = \frac{-0.75}{s+1000} + \frac{2.25}{s+3000}$$

$$v_{zi}(t) = \left[-0.75e^{-1000t} + 2.25e^{-3000t}\right]u(t)$$

What are the natural and forced responses?

Features of s-domain cct analysis

The response transform of a finite-dimensional, lumped-parameter linear cct with input being a sum of exponentials is a rational function and its inverse Laplace Transform is a sum of exponentials

The exponential modes are given by the poles of the response transform

Because the response is real, the poles are either real or occur in complex conjugate pairs

The natural modes are the zeros of the cct determinant and lead to the natural response

The forced poles are the poles of the input transform and lead to the forced response

Features of s-domain cct analysis

- A cct is **stable** if all of its poles are located in the open left half of the complex s-plane
  - A key property of a system
  - Stability: the natural response dies away as $t \rightarrow \infty$
  - Bounded inputs yield bounded outputs
- A cct composed of Rs, Cs and Ls will be at worst marginally stable
  - With Rs in the right place it will be stable
    - $Z(s)$ and $Y(s)$ both have no poles in $\mathrm{Re}(s)>0$
- Impedances/admittances of RLC ccts are "Positive Real" or energy dissipating