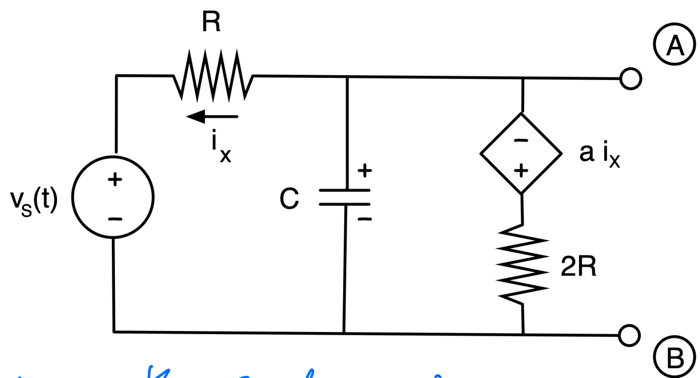
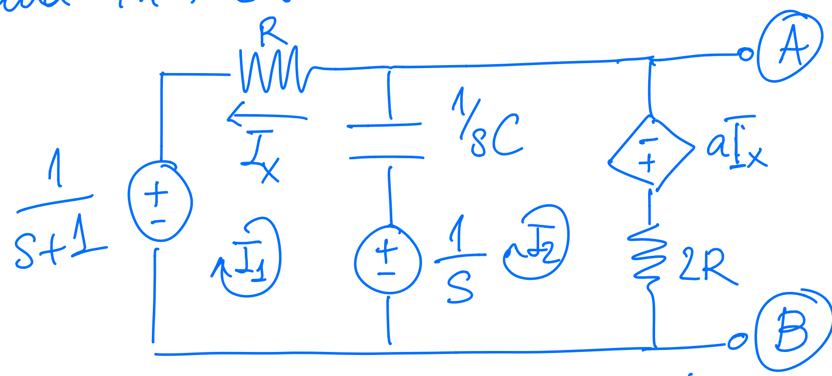
1.– Part I the circuit in the time domain is as follows:



and in the s-domain:



[+1 point for correct initial condition; +1 point for correct overall circuit]

(where we have used $V_s(t) = e^{-t}u(t)$ and $V_c(0) = 1$).

Part II We use mesh-current analysis to find the mesh currents.

KVL @ mesh 1,

$$R\, I_1(s) + \frac{1}{sC}\left(I_1(s) - I_2(s)\right) + \frac{1}{s} - \frac{1}{s+1} = 0$$

[+0.5 point]

KVL @ mesh 2,

$$-aI_x + 2R\,I_2(s) - \frac{1}{s} + \frac{1}{sC}\left(I_2(s) - I_1(s)\right) = 0$$

[+0.5 point]

We need to take care of the dependent source. Note

$$I_x = -I_1.$$

[+1 point]

Substituting and solving,

$$\begin{pmatrix} R + 1/sC & -1/sC \\ a - 1/sC & 2R + \frac{1}{sC} \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \frac{-1}{s(s+1)} \\ \frac{1}{s} \end{pmatrix}$$

$$\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \frac{RCs+1}{sC} & -\frac{1}{sC} \\ \frac{aCs-1}{sC} & \frac{2RCs+1}{sC} \end{pmatrix}^{-1} \begin{pmatrix} \frac{-1}{s(s+1)} \\ 1/s \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1-2CR}{(1+s)(a+3R+2CR^2s)} \\ \frac{1+aC+CR(1+s)}{(1+s)(a+3R+2CR^2s)} \end{pmatrix}$$

Therefore

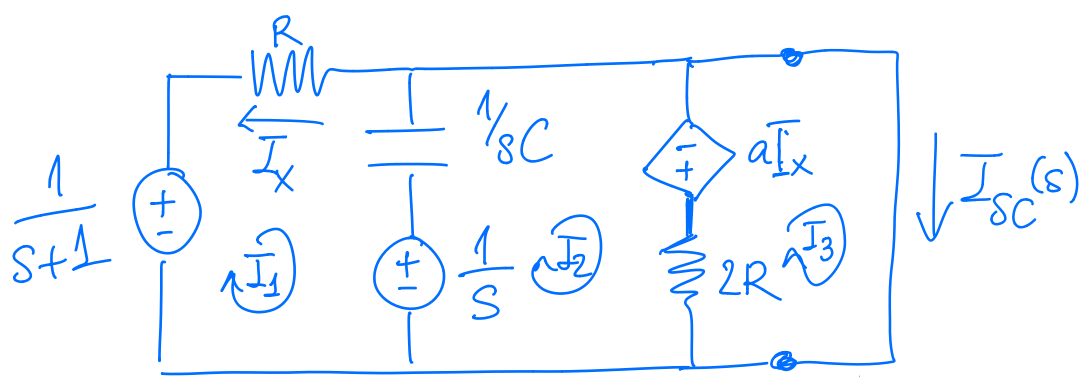
$$V_{AB}(s) = -aI_x + 2R\, I_2(s) = a\, I_1(s) + 2R\, I_2(s)$$

$$= \frac{a + 2R + 2CR^2(1+s)}{(1+s)(a+3R+2CR^2 s)}$$

[+1 point]

Part III

Let's redraw the circuit with the terminals Ⓐ & Ⓑ shorted.



We follow a similar reasoning as in the previous part.

KVL @ mesh 1 is the same as before. [+0.5 point]

KVL @ mesh 2 :

$$-aI_x + 2R(I_2(s) - I_3(s)) - \frac{1}{s} + \frac{1}{sC}(I_2(s) - I_1(s)) = 0$$

[+0.5 point]

KVL @ mesh 3:

$$2R(I_3(s)-I_2(s)) + aI_x = 0$$ [+0.5 point]

Dependent source: $I_x = -I_1(s)$ [+0.5 point]

In matrix form

$$\begin{pmatrix} R+\frac{1}{sC} & -\frac{1}{sC} & 0 \\ a-\frac{1}{sC} & 2R+\frac{1}{sC} & -2R \\ -a & -2R & 2R \end{pmatrix}\begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix} = \begin{pmatrix} \frac{-1}{s(s+1)} \\ \frac{1}{s} \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{R(1+s)} \\ \frac{1+CR+CRs}{R(1+s)} \\ \frac{a+2R+2CR^2(1+s)}{2R^2(1+s)} \end{pmatrix}$$

Hence $I_{sc}(s) = I_3(s) = \dfrac{a+2R+2CR^2(1+s)}{2R^2(1+s)}$ [+1 point]

Part IV

From Parts II and III, we have

$$Z_N(s) = \frac{V_{OC}(s)}{I_{SC}(s)} = \frac{V_{AB}(s)}{I_{SC}(s)} =$$

$$= \frac{a + 2R + 2CR^2(1+s)}{(1+s)(a+3R+2CR^2 s)} \cdot \frac{2R^2(1+s)}{a+2R+2CR^2(1+s)}$$

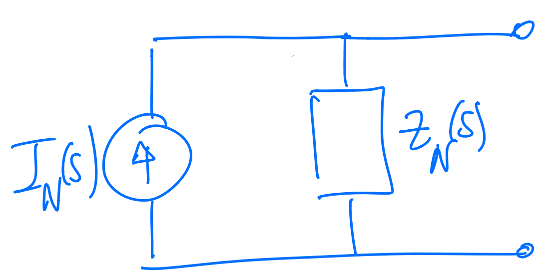
$$= \frac{2R^2}{a+3R+2CR^2 s}$$ [+1 point]

Also,

$$I_N(s) = I_{SC}(s) = \frac{a+2R+2CR^2(1+s)}{2R^2(1+s)}$$

[+1 point]

Hence



Part V

From Part II, we have

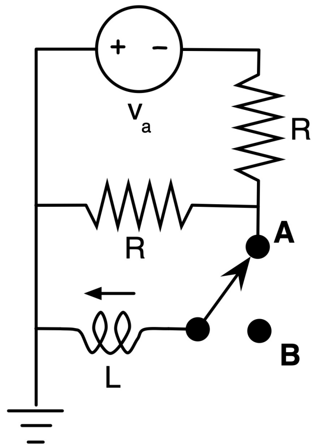
$$V_C(s) = \frac{1}{sC}\left(I_1(s) - I_2(s)\right) + \frac{1}{s} =$$

[+1 extra point for correct expression of capacitor voltage transform]

$$= \frac{a + 2R + 2CR^2(1+s)}{(1+s)(a+3R+2CR^2 s)} = V_{AB}(s)$$

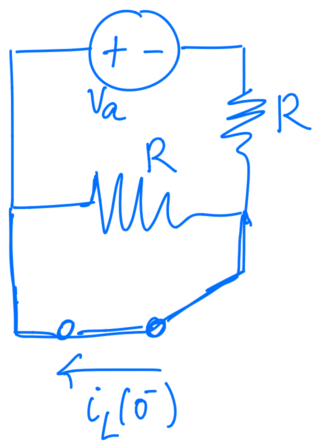
[+1 extra point]

2._ Part I



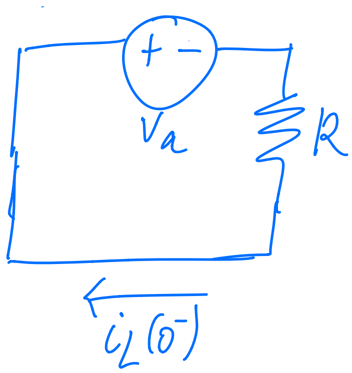
Under DC excitation, we know the inductor behaves as a short circuit. Therefore, we have

[+1 point]



The resistor in the middle does not play any role (since it is in parallel with a short circuit). Hence we can instead draw

[+1 point]

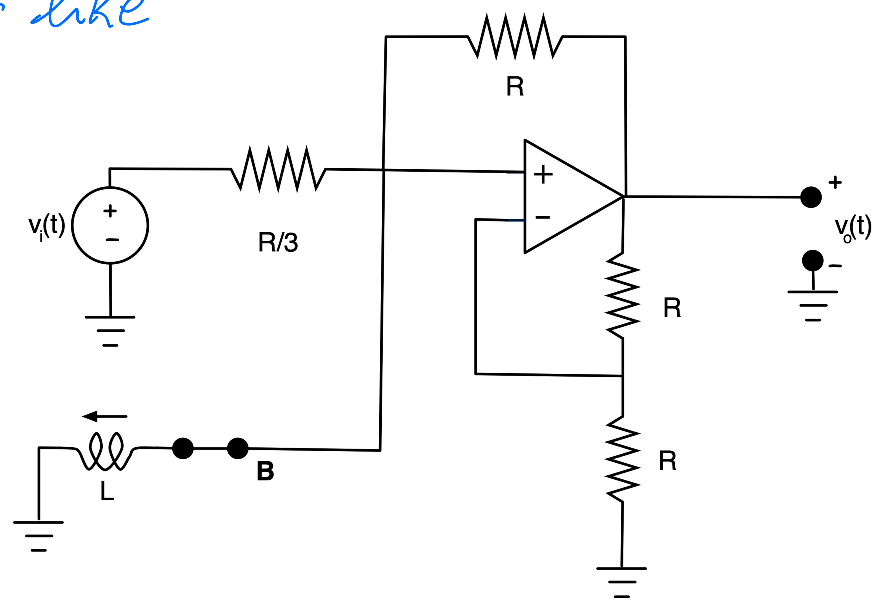


Therefore,

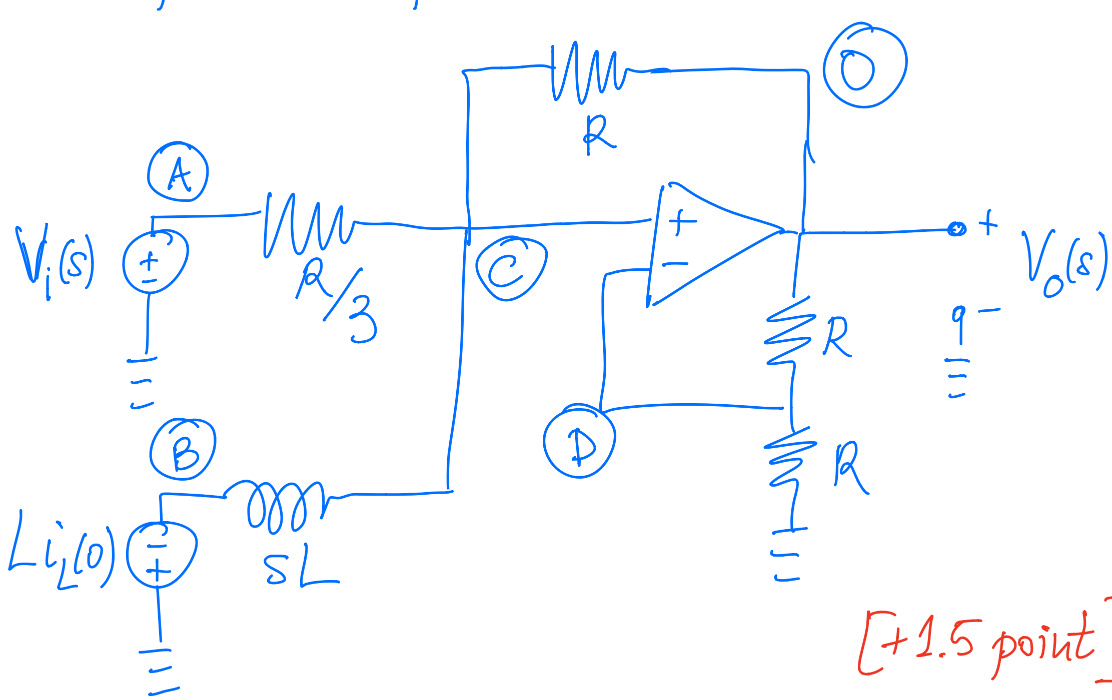
$$i_L(0^-) = \frac{-V_a}{R}.$$

Part II

In the time domain, the circuit to analyze looks like



We first transform it to the s-domain.



[+1.5 point]

Next, we use nodal analysis to find the output response transform.

$$V_A = V_i(s)$$ [+0.5 point]

$$V_B = -L\, i_L(0) = +\frac{L V_A}{R}$$ [+0.5 point]

KCL @ node C (here $G = \frac{1}{R}$)

$$3G(V_C - V_A) + \frac{1}{sL}(V_C - V_B) + G(V_C - V_0) = 0$$ [+0.5 point]

KCL @ node D

$$G(V_D - V_0) + G(V_D) = 0$$ [+0.5 point]

Ideal op-amp conditions

$$V_C = V_D$$ [+0.5 point]

Solving for the unknowns, we set

$$V_0 = 2V_D = 2V_C$$

$$0 = 3G\left(V_C - V_i(s)\right) + \frac{1}{sL}\left(V_C - LGV_A\right) + G\left(V_C - 2V_C\right)$$

$$\left(3G + \frac{1}{sL} - G\right) V_C = 3G\,V_i(s) + \frac{1}{s} G V_A$$

$$\left(\frac{R}{sL} + 2\right) V_C = 3V_i(s) + \frac{1}{s} V_A$$

$$V_C(s) = \frac{sL}{R+2sL}\left(3V_i(s) + \frac{1}{s} V_A\right)$$

Finally,

$$V_o(s) = \frac{2sL}{R+2sL}\left(3V_i(s) + \frac{1}{s} V_A\right)$$

[+1 point]

Part III

With the given values, we have

$$V_o(s) = \frac{s}{s+1/2}\left(3\cdot\frac{1}{s+1} + \frac{1}{s}\right) =$$

$$= \frac{s}{s+1/2}\,\frac{3s+s+1}{s(s+1)} = \frac{4s+1}{\left(s+\frac{1}{2}\right)(s+1)}$$

[+1 point]

Using partial fractions,

$$V_o(s) = \frac{A}{s+1/2} + \frac{B}{s+1}$$

By the residue method

$$A = \lim_{s \to -1/2} \left(s+\frac{1}{2}\right) \cdot V_o(s) = \lim_{s \to \cdot} \frac{4s+1}{s+1}$$

$$= \frac{-2+1}{-1/2+1} = -2$$

[+1 point]

$$B = \lim_{s \to -1} (s+1) \cdot V_o(s) = \lim_{s \to -1} \frac{4s+1}{s+1/2}$$

$$= \frac{-4+1}{-1+1/2} = \frac{-3}{-1/2} = 6$$

[+1 point]

Therefore, taking the inverse Laplace transform,

$$V_o(t) = \left(6e^{-t} - 2e^{-1/2 t}\right) u(t)$$

Part IV

The input has a pole at $s=1$. Therefore, the forced response is

$$(V_0)_{fr}(t) = 6e^{-t}u(t)$$ [+0.5 extra point]

The natural response is then

$$(V_0)_{nr}(t) = -2e^{-1/2t}u(t)$$ [+0.5 extra point]

To compute the zero-state response, we discard the initial condition of the inductor

$$(V_0)_{zs}(s) = \frac{s}{s+1/2}\left(\frac{3}{s+1}\right) = \frac{C}{s+1/2} + \frac{D}{s+1}$$

$C=-3$, $D=6$

Therefore

$$(V_0)_{zs}(t) = \left(6e^{-t} - 3e^{-1/2t}\right)u(t)$$ [+0.5 extra point]

Consequently

$$(V_0)_{zi}(t) = e^{-1/2t}u(t)$$ [+0.5 extra point]

3.- Part I

We compute

$$T(j\omega) = \frac{10}{-\omega^2 + j\omega + 1}$$

Therefore,

$$|T(j\omega)| = \frac{10}{\sqrt{(1-\omega^2)^2 + \omega^2}}$$ [+1 point]

$$\angle T(j\omega) = 0 - \arctan \frac{\omega}{1-\omega^2}$$ [+1 point]

Part II

$$|T(j0)| = \frac{10}{\sqrt{1}} = 10$$ [+0.5 point]

$$|T(j\infty)| = \frac{10}{\sqrt{\infty}} = 0$$ [+0.5 point]

$$\angle T(j0) = -\arctan \frac{0}{1} = 0$$ [+0.5 point]

$$\angle T(j\infty) = -\pi$$ [+0.5 point]

Finally, to find the cutoff frequency, we have to compute the maximum value of the gain function $|T(j\omega)|$.

We do so by computing when the denominator of $|T(j\omega)|$ achieves its minimum (since the numerator is constant). To do so, consider $(1-\omega^2)^2+\omega^2 =$
$= 1+\omega^4-\omega^2$ and derive it,

$$\frac{d}{d\omega}\left(1+\omega^4-\omega^2\right) = 4\omega^3-2\omega = 2\omega\left(2\omega^2-1\right)$$

This vanishes at $\omega=0$, $\omega=\pm\frac{1}{\sqrt{2}}$

$\omega=0$ is a maximum and $\omega=\frac{1}{\sqrt{2}}$ is a minimum, where the function takes the value

$$\left.\left(1+\omega^4-\omega^2\right)\right|_{\omega=\frac{1}{\sqrt{2}}} = 1+\frac{1}{4}-\frac{1}{2} = \frac{3}{4}$$

Therefore $T_{max} = \dfrac{10}{\frac{\sqrt{3}}{2}} = \dfrac{20}{\sqrt{3}}$ [+1 point]

Next, we can find the cutoff frequency by solving

$$|T(j\omega_c)| = \frac{T_{max}}{\sqrt{2}} = \frac{20}{\sqrt{6}}$$

$$\| $$

$$\frac{10}{\sqrt{(1-\omega_c^2)^2 + \omega_c^2}}$$

$$\left(\frac{1}{\sqrt{(1-\omega_c^2)^2 + \omega_c^2}}\right)^2 = \left(\frac{2}{\sqrt{6}}\right)^2$$

$$\frac{1}{(1-\omega_c^2)^2 + \omega_c^2} = \frac{4}{6} = \frac{2}{3} \iff$$

$$\frac{3}{2} = 1 + \omega_c^4 - \omega_c^2 \iff \omega_c^4 - \omega_c^2 - \frac{1}{2} = 0$$

$$\omega_c^2 = \frac{1 \pm \sqrt{1 + 4 \cdot 1/2}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

We take the positive root,

$$w_c = \sqrt{\frac{1+\sqrt{3}}{2}} \simeq 1.1688 \quad \text{[+1 point]}$$

[If somebody uses instead $T_{max} \sim 10$ (the passband gain), then give partial credit for calculation of w_c - up to 1 point, instead of 2 points]

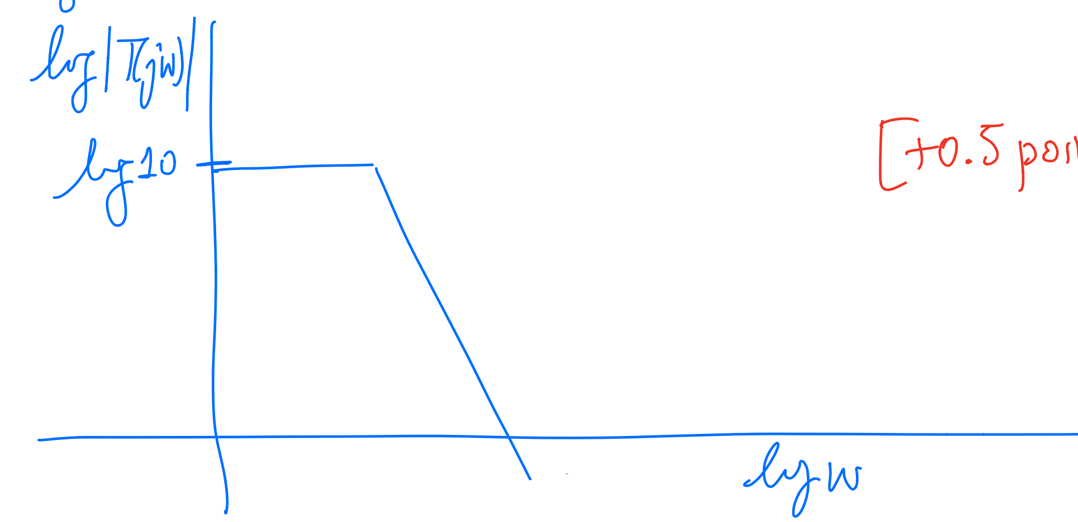
Part III

$$|T(jw)| = \frac{10}{\sqrt{(1-w^2)^2 + w^2}}$$

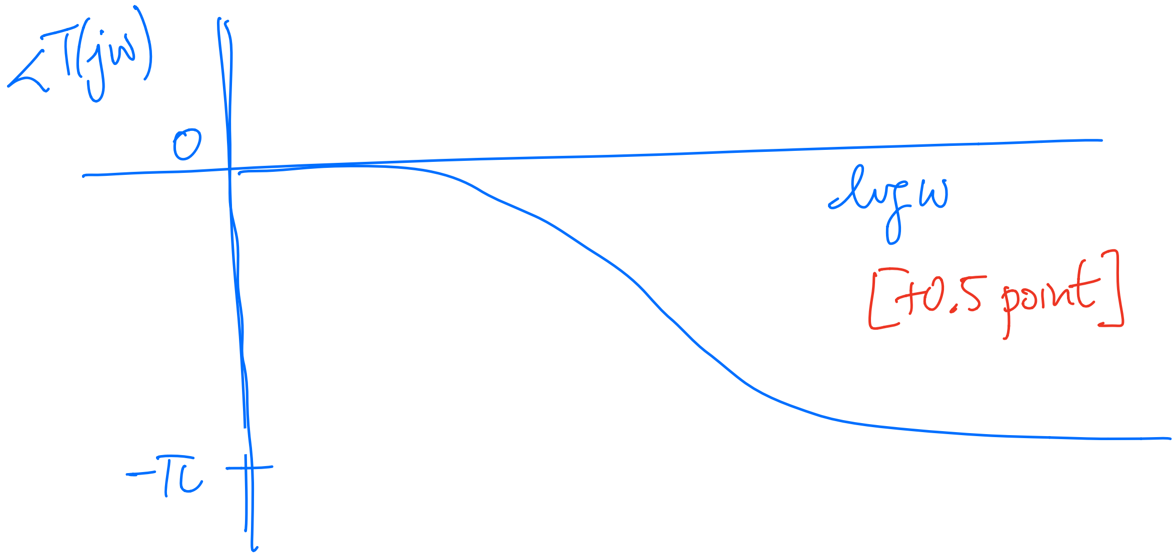
For $w \ll 1$, $|T(jw)| \approx \frac{10}{1} = 10$

For $w \gg 1$, $|T(jw)| \approx \frac{10}{w^2}$

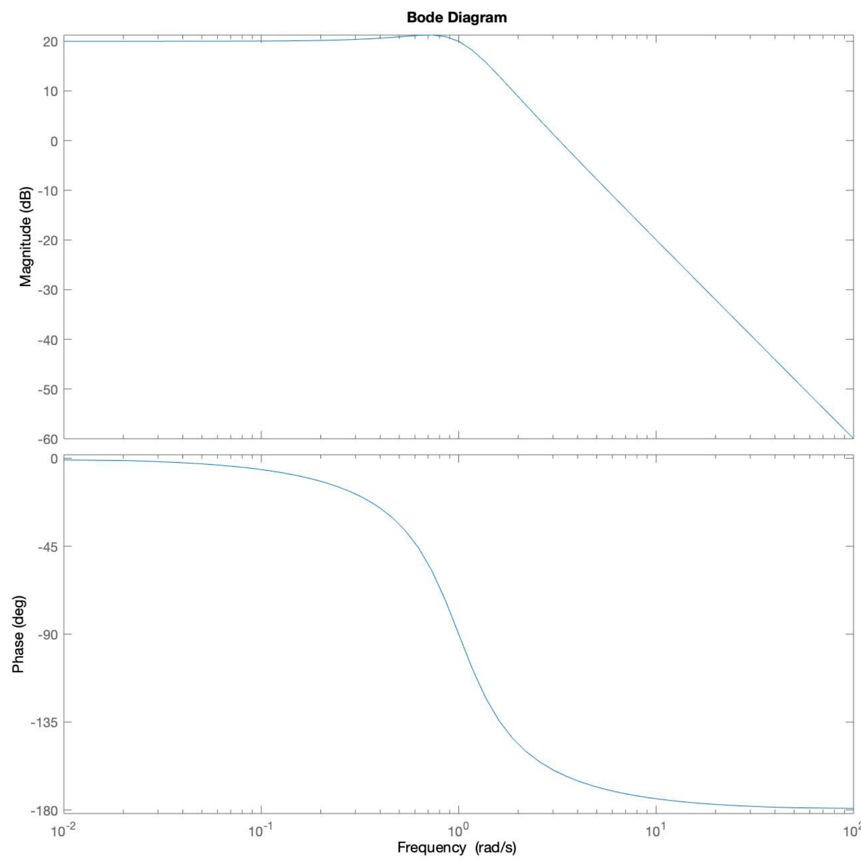
Sketchs can be done in linear scale or logarithmic scale.



[+0.5 point]



[+0.5 point]



This is a low-pass filter.

[+1 point]

Part IV

Given the input

$$v_i(t) = \sqrt{13}\cos\left(2t - \frac{\pi}{2}\right)$$

the steady-state response is

$$v_o^{ss}(t) = \sqrt{13}\,|T(j2)|\cdot\cos\left(2t - \frac{\pi}{2} + \angle T(j2)\right)$$

[+1 point]

$$|T(j2)| = \frac{10}{\sqrt{(1-2^2)^2 + 2^2}} = \frac{10}{\sqrt{13}}$$

$$\angle T(j2) = -\arctan\left(\frac{2}{1-4}\right) =$$

$$= -\left(\pi - \arctan\frac{2}{3}\right) = -\pi + \arctan\frac{2}{3}$$

$$\simeq -2.55359$$

Therefore

$$v_o^{ss}(t) = 10\cos\left(2t - \frac{\pi}{2} + \arctan\frac{2}{3}\right)$$

$$\simeq 10\cos(2t - 4.12439)$$

[+1 point]

4. **Part I**

Block 1 is a voltage divider, so

$$T_1(s) = \frac{2R}{2R + \frac{1}{sC_1}} = \frac{2RC_1 s}{2RC_1 s + 1} = \frac{s}{s + \frac{1}{2RC_1}}$$

[+1 point]

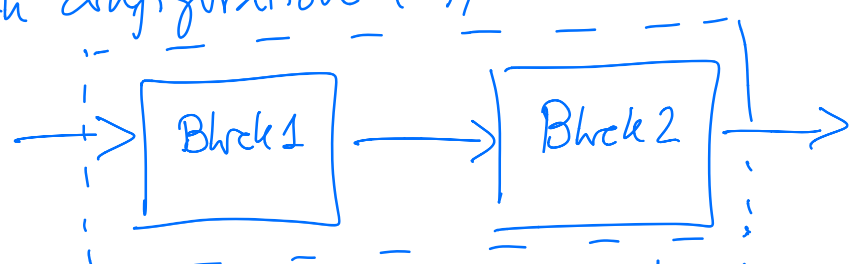
Block 2 is an inverting op-amp, so

$$T_2(s) = -\frac{R + 1/sC_2}{R} = -\frac{RC_2 s + 1}{RC_2 s} = -\frac{s + 1/RC_2}{s}$$

[+1 point]

Part II

In configuration (a), we have

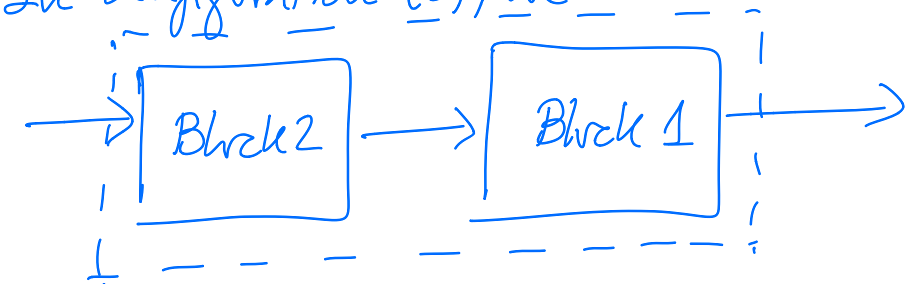


In this case, the chain rule does not apply, because the inverting op-amp (Block 2) will load the voltage divider (Block 1), so

$$T(s) \neq T_1(s) \cdot T_2(s)$$

[+1 point]

In configuration (b), we have



In this case, the chain rule does apply, as the zero output impedance of the op-amp takes care of Block 1 not loading Block 2. Therefore

$$T(s) = T_1(s) \cdot T_2(s)$$

[+1 point]

So, we do not obtain the same transfer functions if we change the order of the blocks, because in configuration (a) there is loading, but not in configuration (b).

Part II

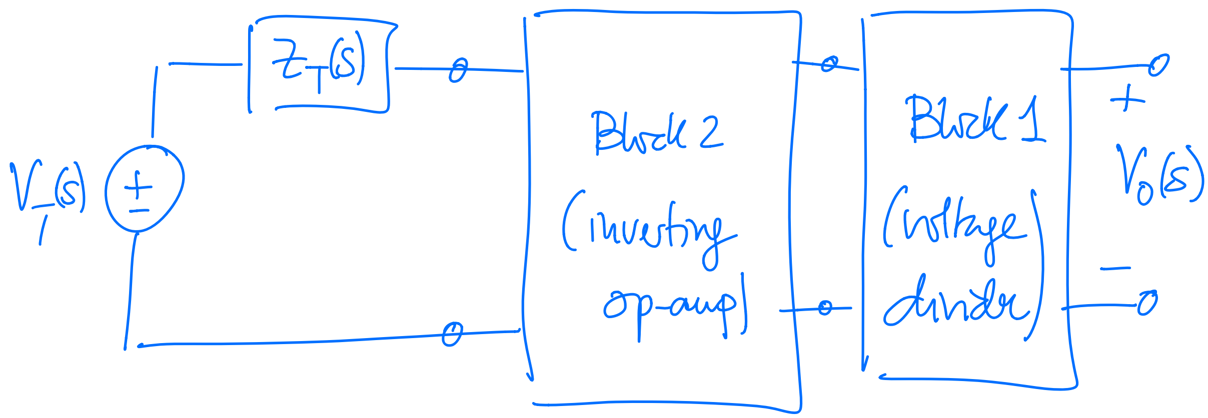
The engineer used configuration (b), whose transfer function is

$$T(s) = T_1(s) \cdot T_2(s) = \frac{\cancel{s}}{s+1/2RC_1} \cdot \left(-\frac{s+1/RC_2}{\cancel{s}}\right) =$$

$$= -\frac{s+1/RC_2}{s+1/2RC_1}$$

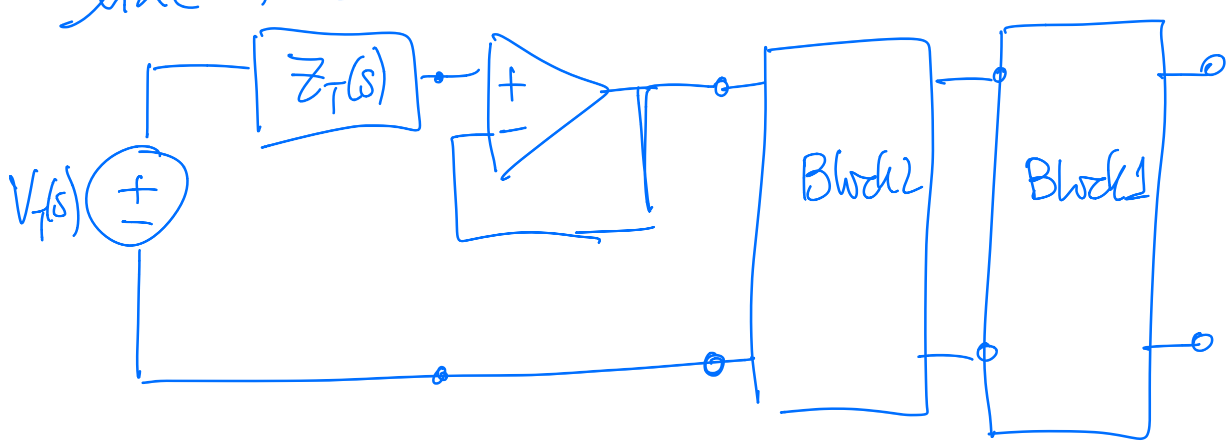
[+1 point]

When we hooked up the source circuit the overall circuit looked like



Note that Block 2 will be loading the source circuit. Unless $Z_T(s) = 0$, this means that $V_o(s) \neq T(s) V_T(s)$. [+1 point]

If we could use one additional op-amp, we could fix this with a voltage follower, like this



The addition of the voltage follower (with transfer function $T_3(s) = 1$) takes care of the loading problem. Now, block 2 does not load the follower (b/c of its 0 output impedance) and the follower does not load the source circuit (b/c of its ∞ input impedance).

[+ 1 point]

Therefore

$$V_o(s) = T_1(s) \cdot T_2(s) \cdot 1 \cdot V_T(s) = T(s) V_T(s)$$

Part IV

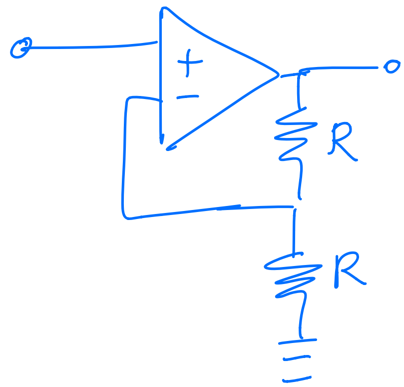
We first verify that the factorization is correct.

$$-2 \cdot \frac{R + \frac{1}{C_2 s}}{2R + \frac{1}{C_1 s}} = -2 \frac{Rs + \frac{1}{C_2}}{2Rs + \frac{1}{C_1}} =$$

$$= -\frac{Rs + 1/C_2}{Rs + \frac{1}{2C_1}} = -\frac{s + 1/C_2 R}{s + 1/2C_1 R} = T(s)$$

[+0.5 point]

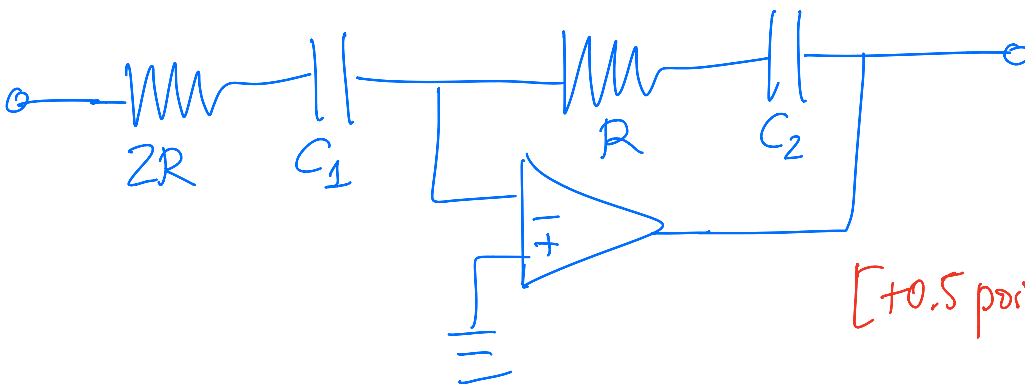
We use the same components ($R, 2R, C_1$ & C_2).
We design a non-inverting op-amp as



whose transfer function is $\tilde{T}_1(s) = 2$

[+0.5 point]

We design an inverting op-amp as

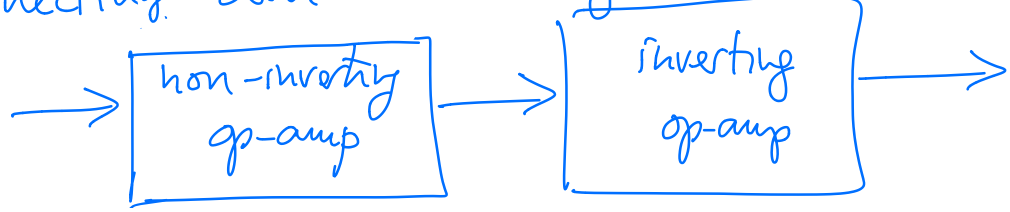


[+0.5 point]

whose transfer function is

$$\tilde{T}_2(s) = -\frac{R + \frac{1}{sC_2}}{2R + \frac{1}{sC_1}}$$

Connecting both circuits gives



There is no loading, hence

[+0.5 point]

$$T(s) = \tilde{T}_1(s) \cdot \tilde{T}_2(s)$$

Moreover, when we connect the source circuit, the ∞-input impedance of the non-inverting op-amp ensures that the source circuit is not loaded. Therefore the output transform is

$$V_o(s) = T(s)\, V_T(s).$$

[+1 point]