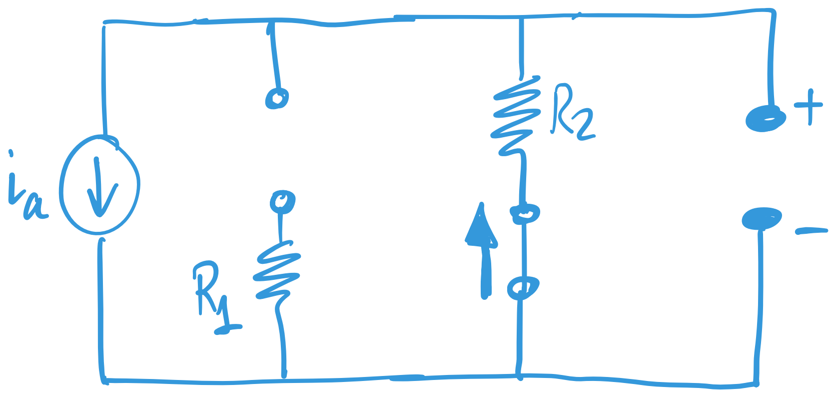
1.– Part I

Under DC excitations, we know the capacitor behaves as an open circuit and the inductor behaves as a short circuit. Therefore we have



[+1 point for correct drawing]

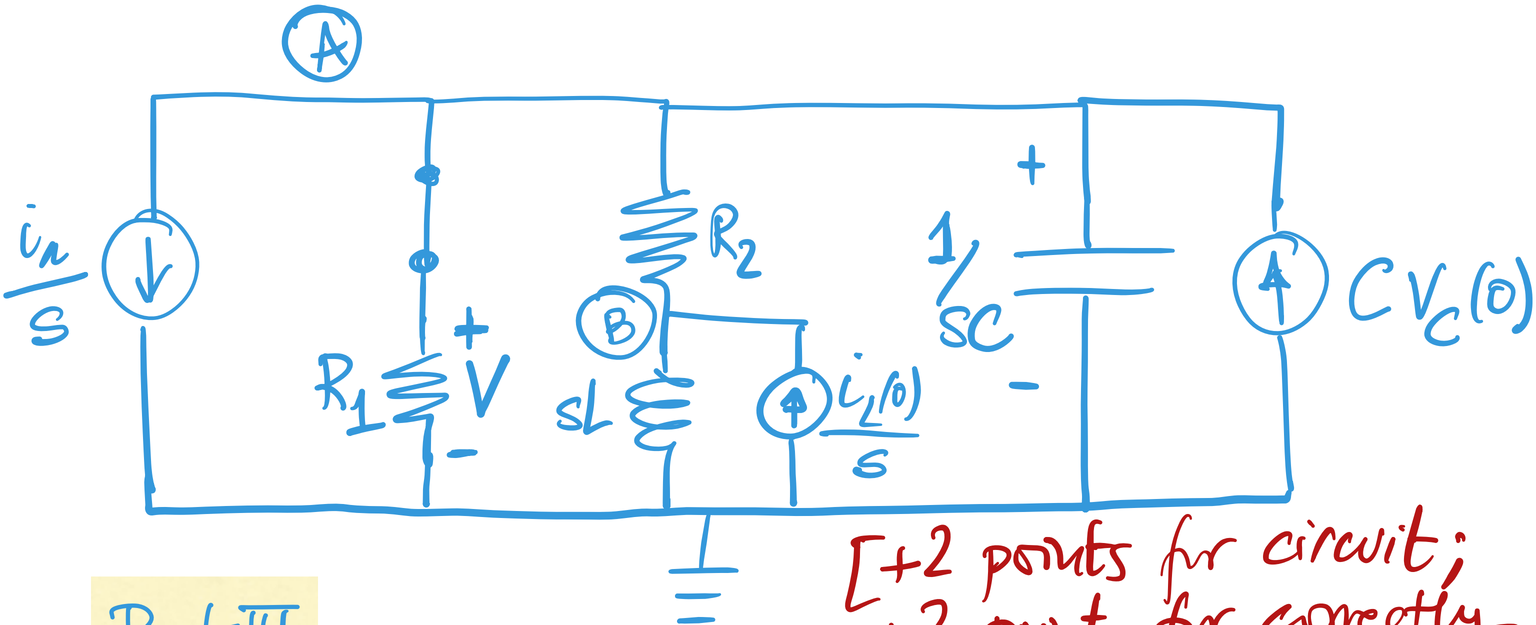
Therefore, we conclude that

$$i_L(0) = i_a$$ [+1 point]

$$V_C(0) = -R_2\, i_a$$ [+1 point]

Part II

We redraw the circuit in the s-domain, using a current source to account for the initial conditions of the inductor and the capacitor.



[+2 points for circuit; +2 point for correctly capturing initial conditions]

Part III

We use nodal analysis to find $V(s)$. Note that $\underbrace{V(s) = V_A}_{[+1\ \text{point}]}$. We write eqs by inspection

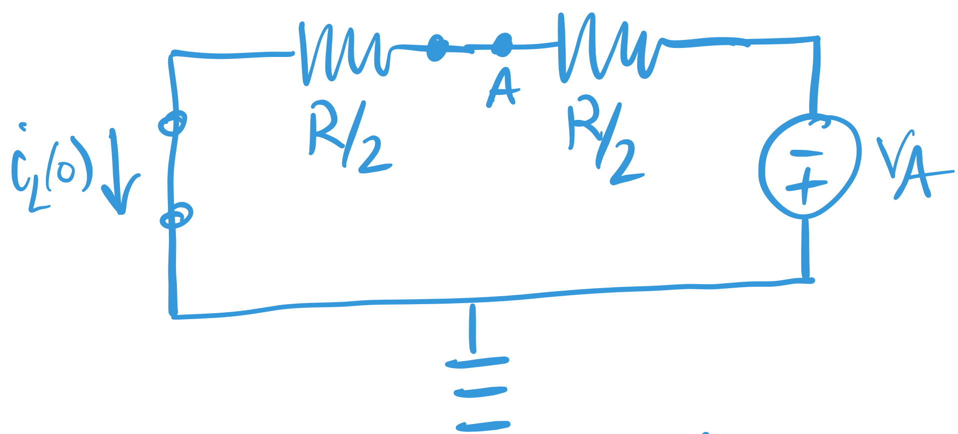
$$\begin{pmatrix} G_1 + G_2 + sC & -G_2 \\ -G_2 & G_2 + \frac{1}{sL} \end{pmatrix} \begin{pmatrix} V_A \\ V_B \end{pmatrix} = \begin{pmatrix} \frac{-i_a}{s} + CV_C(0) \\ \frac{i_L(0)}{s} \end{pmatrix}$$

$$\| $$

[+2 points] $$\begin{pmatrix} \frac{-i_a}{s} - CR_2 i_a \\ \frac{i_a}{s} \end{pmatrix}$$

2.– Part I

To find the initial condition, we substitute the inductor by a short circuit

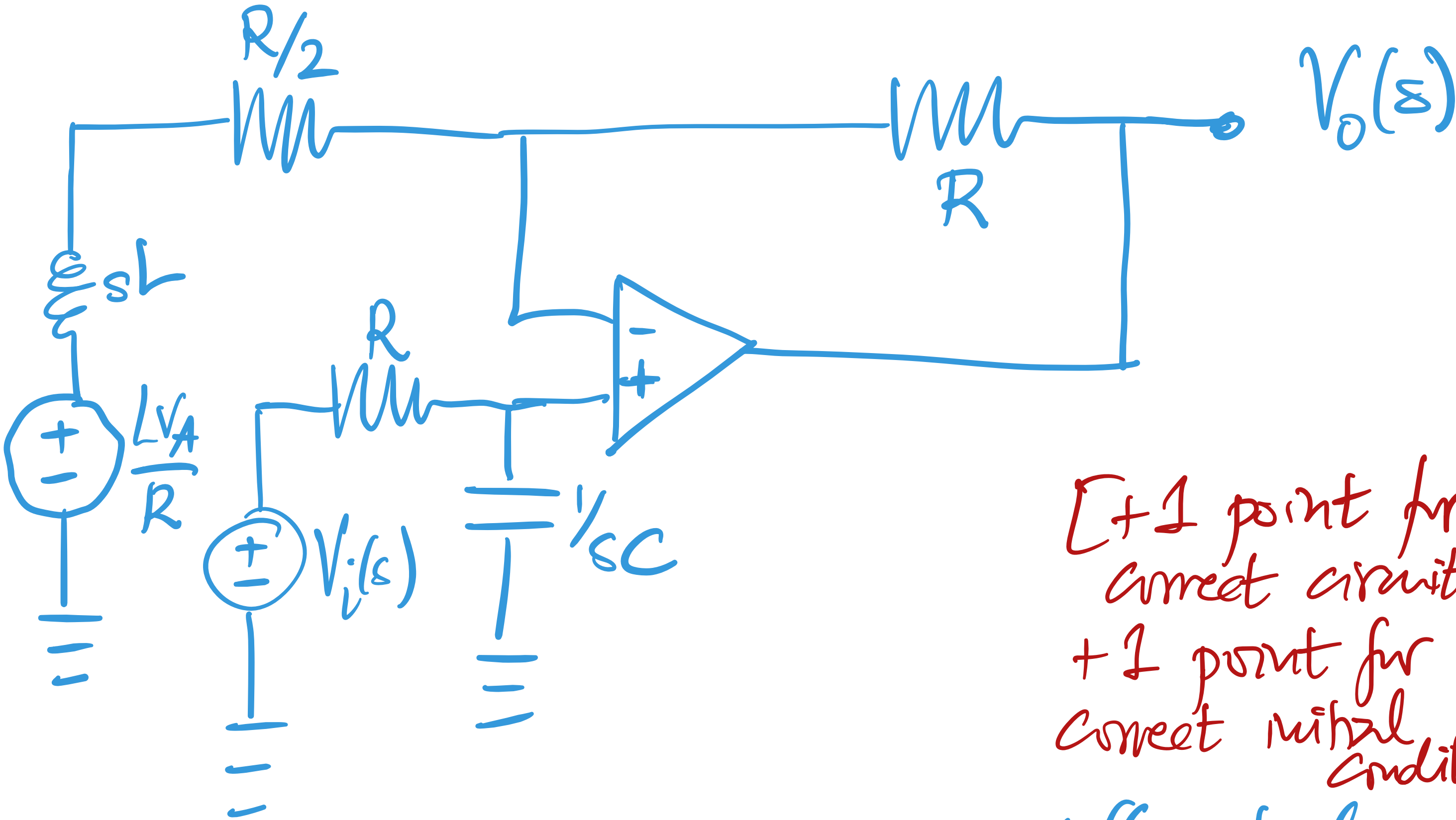


[+1 point]

Therefore $i_L(0) = -\frac{V_A}{R}$

Part II

We add a voltage source, as instructed, to take care of the initial condition of the inductor. We need to carefully take into account the direction indicated for the current to get the polarity right. Also, the initial condition of the capacitor is 0, so no need to worry about it.



[+1 point for correct circuit; +1 point for correct initial condition]

We recognize the circuit as a differential amplifier. Therefore, we can readily [+1 point] compute the output transform as

$$V_o(s) = -\frac{R}{R/2 + sL}\,\frac{LV_A}{R} + \frac{1/sC}{\frac{1}{sC} + R}\cdot\frac{3R/2 + sL}{R/2 + sL}\,V_i(s)$$

$$= \frac{3R + 2sL}{R + 2sL}\cdot\frac{1}{1 + RCs}\,V_i(s) - \frac{2L}{R + 2Ls}\,V_A$$

[+1 point]

Part III

Substituting the values provided, we get

$$V_o(s) = \frac{3 + 2\cdot 10^{-3}s}{1 + 2\cdot 10^{-3}s} \cdot \frac{1}{1+10^{-3}s} \frac{1}{s+1500} - \frac{2\cdot 10^{-3}}{1+2\cdot 10^{-3}s}\cdot 1$$

$$= \frac{3000 + 2s}{1000 + 2s} \cdot \frac{1000}{1000+s} \cdot \frac{1}{s+1500} - \frac{2}{1000+2s}$$

$$= \frac{1000}{(500+s)(1000+s)} - \frac{1}{500+s}$$ [+1 point]

Using partial fractions, we can express

$$V_o(s) = \frac{A}{s+1000} + \frac{B}{s+500} - \frac{1}{s+500}$$

[+1 point]

Using the residue method,

$$A = \lim_{s\to -1000} \frac{1000}{s+500} = -2$$

$$B = \lim_{s \to -500} \frac{1000}{1000+s} = 2$$

Therefore, $V_o(s) = -\frac{2}{s+1000} + \frac{2}{s+500} - \frac{1}{s+500}$

[+1 point]

The output voltage is then

$$V_o(t) = \left(-2e^{-1000t} + e^{-500t}\right) u(t)$$

Part IV

(i) The zero-state response is obtained by zeroing the initial condition of the inductor

$$V_{ozs}(s) = -\frac{2}{s+1000} + \frac{2}{s+500}$$

$$V_{ozs}(t) = 2\left(e^{-500t} - e^{-1000t}\right) u(t)$$

[+0.5 point]

The zero-input response is obtained by zeroing the input $v_i(t)$,

$$V_{ozi}(s) = -\frac{1}{s+500}$$

$$V_{ozi}(t) = -e^{-500t}u(t)$$

[+0.5 point]

(ii)
The input has a pole at $s=-1500$.
However, there is no function in the output transform that has the same pole.
Therefore

$$V_{fr}(s)=0 \quad \& \quad V_{fr}(t)=0$$

[+0.5 point]

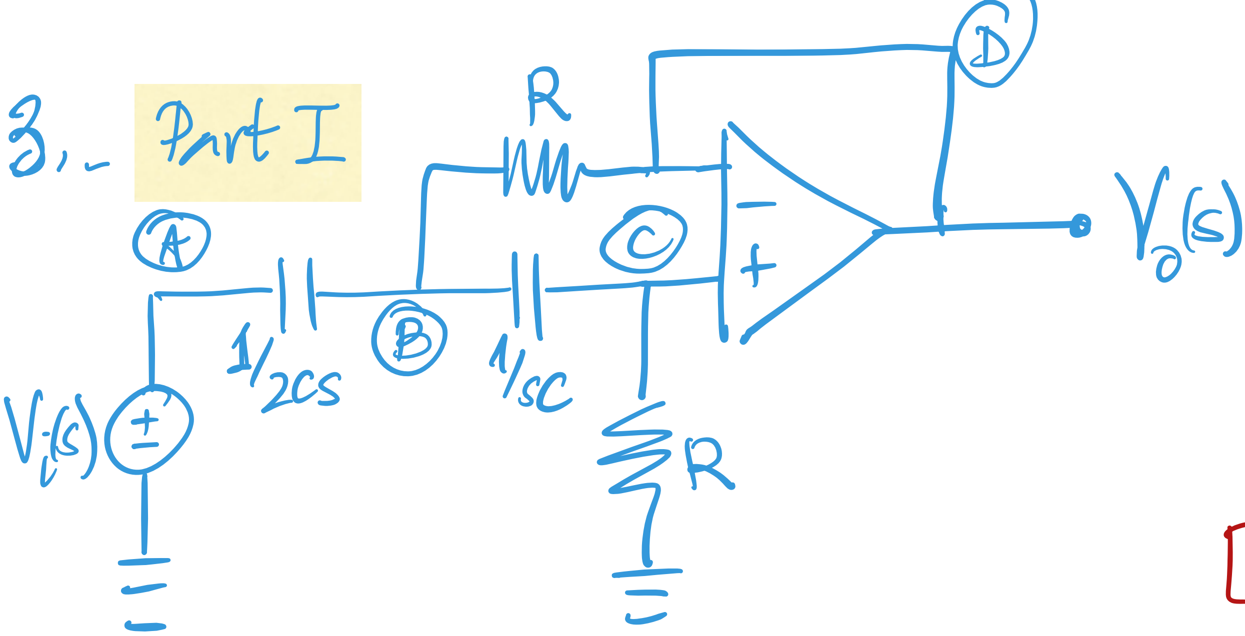
Consequently, the natural response is

$$V_{nr}(s) = V_o(s) \quad \&$$

$$V_{nr}(t) = \left(e^{-500t} - 2e^{-1000t}\right)u(t)$$

[+0.5 point]

3. Part I



[+1 point]

Part II

By looking at the circuit, we know

$$V_A = V_i(s)$$ [+0.5 point]

We know we should not write KCL for the output node of the op-amp. Therefore, we write KCL for nodes B and C, and use ideal op-amp conditions.

KCL @ B

$$2Cs\,(V_B - V_A) + sC\,(V_B - V_C) + \frac{1}{R}(V_B - V_D) = 0$$

[+0.5 point]

KCL @ C

$$sC\,(V_C - V_B) + \frac{1}{R}V_C = 0$$ [+0.5 point]

Ideal op-amp conditions mean that

$$V_C = V_D$$

[+0.5 point]

Finally, from the circuit, we have $V_o(s) = V_D$.

Solving for V_B in KCL@C, we get

$$V_B = \frac{\frac{1}{R}+sC}{sC} V_C = \frac{1+RCs}{RCs} V_C$$

Substituting everything in KCL@B, we get

$$\left(3Cs+\frac{1}{R}\right)\frac{1+RCs}{RCs}V_C - 2Cs\,V_i(s) - \left(sC+\frac{1}{R}\right)V_C = 0$$

$$\underbrace{\left(\frac{3}{R}(1+RCs)+\frac{1+RCs}{R^2Cs} - sC - \frac{1}{R}\right)}_{\displaystyle \frac{3RCs+3R^2C^2s^2+1+\cancel{RCs}-R^2C^2s^2-\cancel{RCs}}{R^2Cs}} V_C = 2Cs\,V_i(s)$$

$$\Longrightarrow \frac{2R^2C^2s^2+3RCs+1}{R^2Cs} V_C = 2Cs\,V_i(s)$$

Hence

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{2R^2C^2s^2}{2R^2C^2s^2 + 3RCs + 1} =$$

$$= \frac{s^2}{s^2 + \frac{3}{2}\frac{1}{RC}s + \frac{1}{2R^2C^2}} =$$

(with $\lambda = \frac{1}{RC}$) $$= \frac{s^2}{s^2 + \frac{3}{2}\lambda s + \frac{1}{2}\lambda^2}$$ [+1 point]

Part III We evaluate at $s = jw$,

$$T(jw) = \frac{-w^2}{-w^2 + \frac{3}{2}\lambda wj + \frac{1}{2}\lambda^2} = \frac{-w^2}{(-w^2 + \frac{1}{2}\lambda^2) + \frac{3}{2}w\lambda j}$$

Therefore

$$|T(jw)| = \frac{w^2}{\sqrt{(-w^2 + \frac{1}{2}\lambda^2)^2 + (\frac{3}{2}w\lambda)^2}}$$ [+1 point]

$$\angle T(j\omega) = \pi - \arctan \frac{\frac{3}{2}\omega}{-\omega^2 + \frac{1}{2}\lambda^2}$$

[+1 point]

Part IV

At $\omega=0$, we have

$$|T(j0)| = \frac{0}{(\frac{1}{2}\lambda^2)^2 + 0} = 0$$

[+0.5 point]

$$\angle T(j0) = \pi - 0 = \pi$$

[+0.5 point]

At $\omega=+\infty$, we have

$$|T(j\infty)| = 1$$

[+0.5 point]

$$\angle T(j\infty = \pi - \pi = 0$$

[+0.5 point]

At $\omega=\lambda$,

$$|T(j\bar{\omega})| = \frac{\lambda^2}{\sqrt{\frac{1}{4}\lambda^4 + \frac{9}{4}\lambda^4}} = \frac{2\cancel{\lambda^2}}{\sqrt{10}\cancel{\lambda^2}} \simeq 0.63$$

[+0.5 point]

Part V

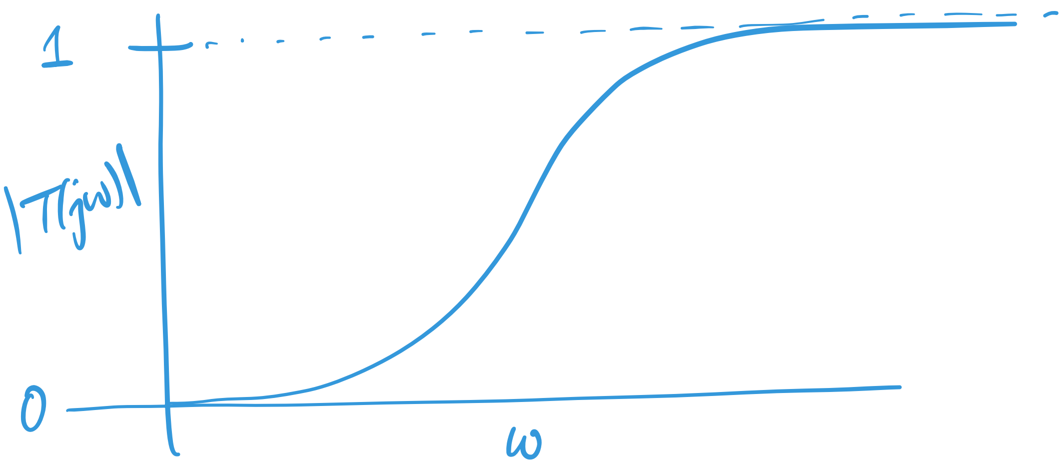
This is a high-pass filter. [+0.5 point]

With $T_{max} = 1$, we have

$$\frac{T_{max}}{\sqrt{2}} \simeq 0.70 > 0.63,$$

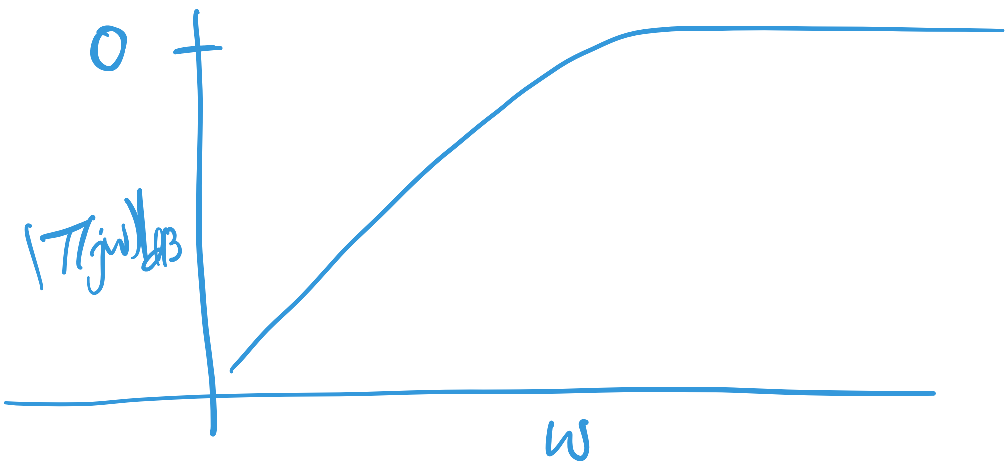
therefore the cut-off frequency w_c is larger than λ. [+0.5 point]

The sketch of the gain is



linear-log

[+0.5 point]
(either plot)



log-log

4.. Part I

Given $T_1(s) = \frac{\pm s}{s+w_1}$ & $T_2(s) = \frac{\pm s}{s+w_2}$,

if we multiply, we obtain

$$T_1(s)\cdot T_2(s) = \frac{s^2}{(s+w_1)(s+w_2)} = \frac{s^2}{s^2+(w_1+w_2)s+w_1w_2}$$

Thus we set

$$w_1+w_2 = \frac{3\lambda}{2} \quad \& \quad w_1w_2 = \frac{\lambda^2}{2}$$ [+1 point]

and solve

$$w_1 + \frac{\lambda^2}{2w_1} = \frac{3\lambda}{2} \Leftrightarrow 2w_1^2+\lambda^2-3\lambda w_1 = 0$$

$$w_1 = \frac{3\lambda \pm \sqrt{9\lambda^2-4\cdot 2\cdot\lambda^2}}{2\cdot 2} = \frac{3\lambda\pm\lambda}{4} = \begin{cases}\lambda \\ \frac{\lambda}{2}\end{cases}$$

Hence $w_2 = \frac{\lambda^2}{2} = \begin{cases}\lambda/2 \\ \lambda\end{cases}$; we select $w_1 = \lambda$, $w_2 = \frac{\lambda}{2}$

[+2 points]

Cut-off freq & gain of T_1 ; $w_c = \lambda$; $T_{max} = 1$

" " " " " T_2 ; $w_c = \frac{\lambda}{2}$; $T_{max} = 1$

[+1 point]

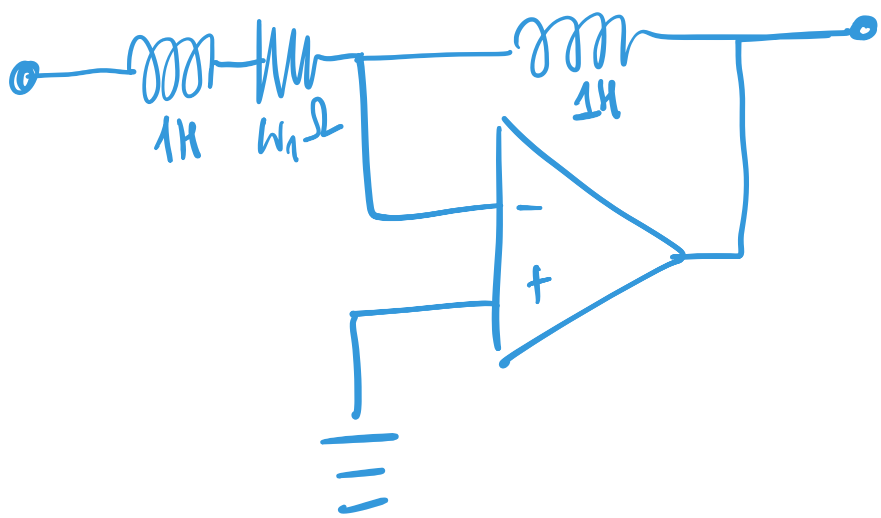
Part II

We use the decomposition

$$T(s) = \underbrace{\frac{-s}{s+w_1}}_{\text{stage 1}} \cdot \underbrace{\frac{-s}{s+w_2}}_{\text{stage 2}}$$

Each function can be realized with an inverting op-amp circuit. For instance,

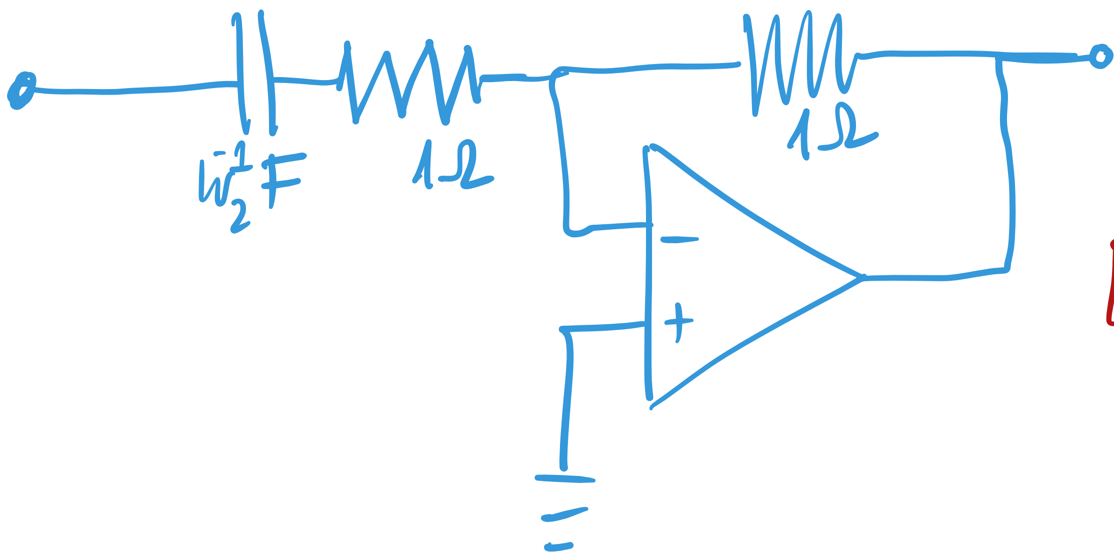
$$T_1(s) = -\frac{s}{s+w_1}$$



[+1 point]

The transfer function $T_2(s)$ can be designed the same way. One can also use

$$T_2(s) = \frac{-s}{s+w_2} = -\frac{1}{1+\frac{w_2}{s}} = -\frac{1}{1+\frac{1}{sw_2^{-1}}}$$



[+1 point]

The connection in series of these two designs leads to $T(s) = T_1(s) \cdot T_2(s)$ as there is no loading, because of the zero output impedance of the op-amp.

[+1 point]

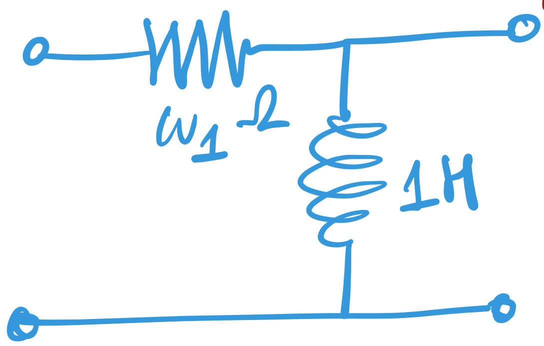
Part III

Our designs of the previous part do not work in this case because we can only use 1 OpAmp. Instead, we rely on the following decomposition

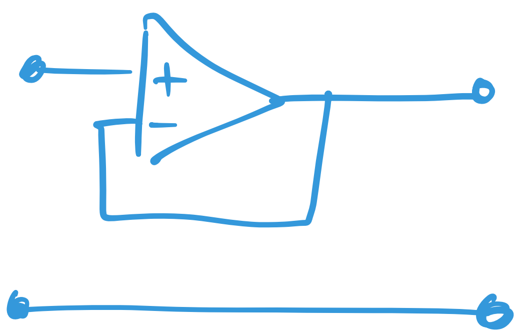
$$T(s) = \underbrace{\frac{s}{s+w_1}}_{\text{stage 1}} \cdot \underbrace{1}_{\text{stage 2}} \cdot \underbrace{\frac{s}{s+w_2}}_{\text{stage 3}}$$

We can make stages 1 & 3 happen w/ voltage dividers, and we can make stage 2 happen w/ a follower. So long as the follower is in between the dividers, there is no loading and the chain rule applies. [+1 point]

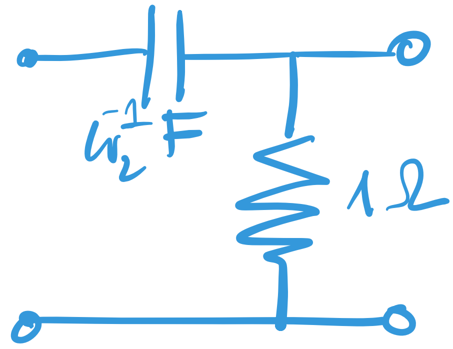
$$T_1(s) = \frac{s}{s+w_1}$$



$$T_{\text{follower}}(s) = 1$$



$$T_2(s) = \frac{s}{s+\omega_2} = \frac{1}{1+\frac{1}{\omega_2^{-1}s}}$$



[+1 point]

Either

$$\rightarrow \boxed{T_1(s)} \rightarrow \boxed{T_{follower}(s)} \rightarrow \boxed{T_2(s)} \rightarrow$$

or

$$\rightarrow \boxed{T_2(s)} \rightarrow \boxed{T_{follower}(s)} \rightarrow \boxed{T_1(s)} \rightarrow$$

will work.

[+1 point]

5.. Part I

Let's start by computing the transfer function

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{R_2}{R_1}$$

With $R_1 = R_2 = 600\Omega$, we obtain $T(s) = -1$

Therefore

$$v_o(t) = -48 - 12\cos(1500t)$$

if the op-amp operates in the linear regime, i.e.,

$$-12 \leq v_o(t) \leq 12$$

However, note that $v_o(t) < -12$, and therefore the output is actually saturated,

[+1 extra point]

$$v_o(t) = -12V$$

hence no 1500 rad/s tone can be heard.

Part II

With a potentiometer / variable resistor, we instead have

$$T(s) = -\frac{R_2}{600}$$

The output (if operating in the linear regime) would be

$$V_0(t) = -\frac{R_2}{600} \cdot 48 - \frac{R_2}{600} \cdot 12 \cos(1500)t$$

We need $-12 \leq V_0(t) \leq 12$

Let's find out the value of R_2 that makes $v_0(t)$ not go below $-12V$.

$$-12 = -\frac{R_2}{600} \cdot 48 - \frac{R_2}{600} 12 \cdot 1$$

(since $1 \leq \cos(1500)t \leq 1$)

$$\frac{R_2}{600}(48+12) = 12 \Leftrightarrow \frac{R_2 \, 60}{600} = 12 \Leftrightarrow$$

$R_2 = 120\Omega$

[+1 extra point]

The corresponding output is

$$v_o(t) = -\frac{48}{5} - \frac{12}{5}\cos(1500t) =$$

$$= -9.6 - 2.4\cos(1500t)$$

The amplitude of the tone is 2.4V.

[+1 extra point]

Part III

The transfer function for circuit 2 is

$$T(s) = \frac{-R_2}{R_1 + \frac{1}{sC}} = \frac{-600}{600 + \frac{1}{sC}} =$$

(note this is a high-pass filter) $$= \frac{-s}{s + \frac{1}{600C}}$$ $$\left(w_c = \frac{1}{600C} \right)$$

The gain function is $|T(jw)| = \dfrac{w}{\sqrt{w^2 + w_c^2}}$

The phase function is $\angle T(jw) = \dfrac{3\pi}{2} - \arctan\dfrac{w}{w_c}$

Therefore,

$$v_o^{ss}(t) = 48 \cdot |T(j0)| \cos(\angle T(j0)) +$$

$$12 \cdot |T(j1500)| \cdot \cos(1500t + \angle T(j1500))$$

$$= 48 \cdot 0 + 12 \cdot \frac{\omega}{\sqrt{\omega^2+\omega_c^2}} \cdot \cos(1500t + \angle T(j1500))$$

[+1 extra point]

Since $\frac{\omega}{\sqrt{\omega^2+\omega_c^2}} \leq 1$, we deduce that

$$-12 \leq v_o^{ss}(t) \leq 12$$

So the op-amp in circuit 2 never saturates and operates in the linear regime. The inclusion of the capacitor effectively blocks the DC component of the input, allowing the tone to be "heard" at the speaker.

[+1 extra point]